Conditionals and modularity in general logics

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Contents

Chapter 1

Introduction

Unless said otherwise, we work in propositional logic.

1.1 The main subjects of this book

This text centers around the following main subjects:

(1) The concept of modularity and independence in

- classical logic
- non-monotonic and other non-classical logic

and the consequences on

- (syntactic and semantic) interpolation and
- language change

In particular, we will show the connection between interpolation for non-monotonic logic and manipulation of an abstract notion of size.

Modularity is, for us, and essentially, the ability to put partial results achieved independently together for a global result.

(2) A uniform picture of conditionals, including

- many-valued logics, and
- structure on the language elements themselves (in contrast to structure on the model set) and on the truth value set

(3) Neighbourhood semantics, their connection to independence, and their common points and differences for various logics, e.g.,

- for defaults and deontic logic

- for the limit version of preferential logics
- and for general approximation.

These subjects are not always isolated from one another, and we will sometimes have to go back and forth between them. For instance, a structure on the language elements can itself be given in a modular way, and this then has influence on the modularity of the structure on the model set. Independence seems to be a core idea of logic, as logic is supposed to give the basics of reasoning, so we will not assume any "hidden" connections - everything which is not made explicitely otherwise, will be assumed to be independent.

The main connections between the concepts investigate in this book are illustrated by Diagram 1.1.1 (page 13). The left hand side concerns non-monotonic logic, the right hand side monotonic or classical logic. The upper part concerns mainly semantics, the lower part syntax.

Independence is at the core. It can be generated by Hamming distances and relations, and can be influenced by stuctures on the language and the truth values. Independence is expressed by the very definition of semantics in classical logic - a formula depends only on the value of the variables occurring in the formula - and by suitable multiplication laws for abstract size in the non-monotonic case. Essentially, by these laws, a product of two sets is big iff the components are big. In neighbourhood semantics, independence is expressed by independent satisfiability of "close" or "good" along several criteria.

Semantical interpolation is the existence of "simple" model sets X between (in the two-valued case: between by set inclusion) the left and the right hand model sets: $\phi \models \psi$ results in $M(\phi) \subseteq X \subseteq M(\psi)$. "Simple" means here that X is restricted only in the parts where both $M(\phi)$ and $M(\psi)$ are restricted, and otherwise the full product, i.e., all truth values may be assumed. A - for the authors - surprising result was that monotonic logic and antitonic logic *always* have semantical interpolation. This results from the independent definition of validity. The same is not necessarily true for full non-monotonic logic (note, that we have $\phi \mid\!\sim \psi$ iff $M(\mu(\phi)) \subseteq M(\psi)$, where $M(\mu(\phi)) \subseteq M(\phi)$, so we have a combined downward and upward movement: $M(\phi)$ - $M(\mu(\phi))$ - $M(\psi)$). The reason is that abstract size (the sets of "normal" or "important" elements) need not be defined in an independent way.

Semantical interpolation results also in syntactic interpolation, if the laguage and the operators are sufficiently rich to express these semantical interpolants. This holds both for monotonic and non-monotonic logic.

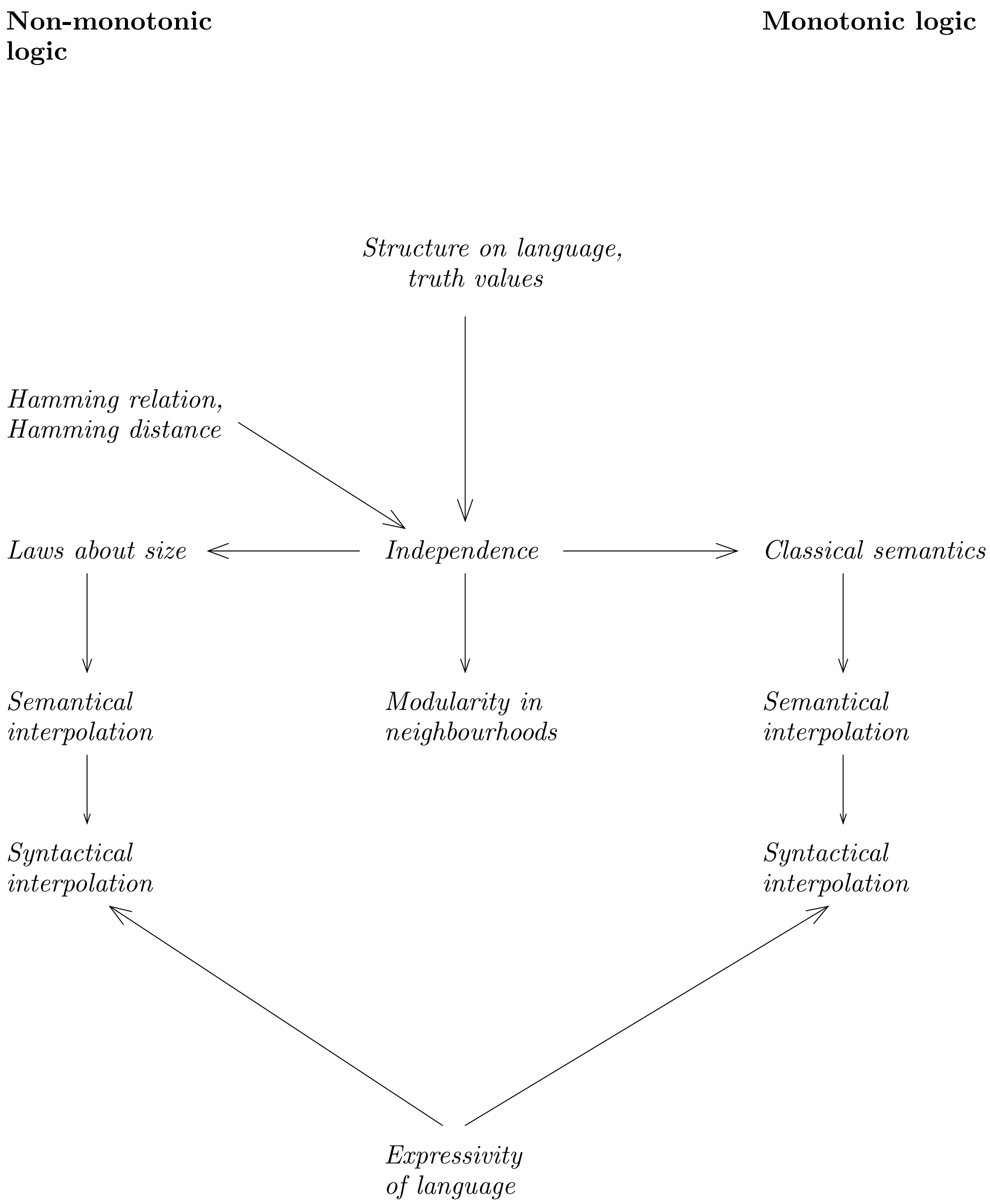
Many aspects of independence can also be illustrated by language change. (We did not include this in the diagram, as it might have become too complicated to read otherwise.)

First, consider classical logic. Let $\phi := p$, p a propositional language. Consider first L, with propositional variable just p, and then L', with variables p, q. As long as we know the value of p in a model m, it is irrelevant whether m is also defined for q or not, and if it is, what value q has. This sounds utterly trivial, but it is not, it is a profound conceptual property, and the whole idea of tables of truth values is based on it. (Unfortunately, this is usually not told to beginners, so they learn a technique (truth value tables) without understanding that there *is* a problem, and how it is solved.) This independence property is all we need to show semantical interpolation, also in many-valued logics, see Proposition 4.2.3 (page 134).

In non-monotonic logic, we do *not* have a priori such an independence property. Consider again L and L' as above, and a preferential structure. In L, the $p-$model might be better than the $\neg p-$model, but this does not mean that in L', $p-$models are better than $\neg p-$models. So we might have, e.g., $\langle p\rangle \prec \langle\neg p\rangle$ in $M(L)$, but $\langle\neg p, q\rangle \prec \langle\neg p\neg q\rangle \prec \langle p, q\rangle \prec \langle p\neg q\rangle$, to give an example. So, in L, we have $TRUE \mid\!\sim p$, in L', we have $TRUE \mid\!\sim \neg p$. (In terms of abstract size, $\{\langle p\rangle\}$ is the smallest big subset of $M(L)$, and $\{\langle\neg p, q\rangle\}$ the smallest big subset of $M(L')$, see Chapter 5 (page 165).) Thus, language matters. Of course, we can impose additional restrictions on the various relations $\prec$ in L and L', e.g., that p has precedence, like if $p \prec \neg p$ in L, then, no matter what the $q-$value is in L', $p-$models will be better. But this is, again, conceptually non-trivial, and here it might be false, contrary to the classical case above.

We did not pursue this point in detail, yet it is very important, and should be considered in further research - and borne in mind. For some remarks, see Section 5.2.2.2 (page 176) and Table 5.3 (page 191).

Diagram 1.1.1



Connections between main concepts

We now give a short introduction to these main subjects.

1.2 Overview of this introduction

In the next sections, we give an introduction to the following chapters of the book. In Section 1.7 (page 32), we try to give an abstract definition of independence and modularity (limited to our purposes). We conclude this chapter with remarks on where we used previously published material (basic definitions etc.), and acknowledgements.

1.3 Basic definitions

This chapter is relatively long, as we use a number of more or less involved concepts, which have to be made precise. In addition, we also want to put our work a bit more in perspective, and make it self-contained, for the convenience of the reader. Most of the material of this chapter (unless marked as "new") was published previously, see [Sch04], [GS08b], [GS08c], [GS09a], and [GS08f].

We begin with basic algebraic and logical definitions, including in particular many laws of non-monotonic logics, in their syntactic and semantic variants, showing also the connections between both sides, see Definition 2.2.6 (page 44) and the tables Table 2.1 (page 52) and Table 2.2 (page 53).

It seems to be a little known result that even the classical operators permit an unusual interpretation in the infinite case, but we claim no originality, see Example 2.2.1 (page 44).

We would like to emphasize the importance of the definability preservation (dp) property. In the infinite case, not all model sets X are definable, i.e., there need not necessarily be some formula ϕ or theory T such that $X = M(\phi)$ - the models of ϕ - or $X = M(T)$ - the models of T. It is by no means evident that a model choice function μ, applied to a definable model set, gives us back again a definable model set (is definability preserving, or in short, dp). If μ does not have this property, some representation results will not hold, which hold if μ is dp, and representation results become much more complicated, see [Sch04] for positive and for impossibility results. In our present context, definability is again an important concept. Even if we have semantic interpolation, if language and operators are not strong enough, we cannot define the semantic interpolants, so we have semantic, but not syntactic interpolation. Examples are found in finite Goedel logics, see Section 4.4 (page 146). New operators guaranteeing the definability of particularly interesting, "universal" interpolants, see Definition 4.3.1 (page 136), are discussed in Section 4.3 (page 136) . They are intricately related to the existence of conjunctive and disjunctive normal forms, as discussed in Section 4.3.3 (page 137).

We conclude this part with a - to our knowledge - unpublished result that we can define only countably many inconsistent formulas, see Example 2.2.2 (page 46). (The question is due to D.Makinson.)

We then give a detailed introduction into the basic concepts of many-valued logics, again, as readers might not be so familiar with the generalizations from 2-valued to many-valued logic. In particular, the nice correspondence between 2-valued functions and sets does not hold any more, so we have to work with arbitrary functions, which give values to models. We have to re-define what a definable model "set" is, and what semantical interpolation means for many-valued logic. A formula ϕ defines such a model value function f_ϕ, and we call a model value function f definable iff there is some formula ϕ such that $f = f_\phi$. Table 2.3 (page 54) gives an overview.

We then give an introduction to preferential structures and the logic they define. These structures are among the best examined semantics for non-monotonic logics, and Chapter 5 (page 165) is also based on the investigation of such structures. We first introduce the minimal variant, and then the limit variant. The first variant is the ususal one, the second is needed to deal with cases where there are no minimal models, due to infinite descending chains. (The first variant was introduced by Y.Shoham in [Sho87b], the second variant by P.Siegel et al. in [BS85]. It should, however, be emphasized, that preferential models were introduced as a semantics for deontic logic long before they were investigated as a semantics for non-monotonic logic, see [Han69]). The limit variant was further investigated in [Sch04], and we refer the reader there for representation and impossibility results. An overview of representation results for the minimal variant is given in Table 2.4 (page 59).

We introduce a new concept in this section on preferential structures, "bubble structures", which, we think, present a useful tool for abstraction, and are a semantic variant of independence in preferential structures. Here, we have a global preferential structure between subsets ("bubbles") of the model set, and a fine scale structure inside those subsets. Seen from the outside, all elements of a bubble behave the same way, so the whole set can be treated as one element, on the inside, we see a finer structure.

Moreover, new material on many-valued preferential structures is included.

We then go into details in the section on IBRS, introduced by D.Gabbay, see [Gab04], and further investigated in [GS08b] and [GS08f], as they are not so much common knowledge. We also discuss here if and how the limit version of preferential structures might be applied to reactive structures.

We then present theory revision, as introduced by Alchorron, Gardenfors, and Makinson, see [AGM85]. Again, we also build on previous results by the authors, when we discuss distance based revision, introduced by Lehmann, Magidor, and Schlechta, see [LMS95], [LMS01], and elaborated in [Sch04]. We also include a short paragraph on new material for theory revision based on many-valued logic.

1.4 Towards a uniform picture of conditionals

In large parts, this chapter should rather be seen more as a sketch for future work, than a fully elaborated theory.

1.4.1 Discussion and classification

It seems difficult to say what is not a conditional. The word "condition" suggests something like "if ..., then ...", but as the condition might be hidden in an underlying structure, and not expressed in the object language, a conditional might also be an unary operator, e.g., we may read the consequence relation $\mid\sim$ of preferential structure as "under the condition of normality".

Moreover, as shown at the beginning of Section 3.1 (page 97), Example 3.1.1 (page 97), it seems that one can define new conditionals ad libitum, binary, ternary, etc.

Thus, the best seems to be to say that a conditional is just any operator. Negation, conjunction, etc., are then included, but excluded from the discussion, as we know them well.

The classical connectives have a semantics in the boolean set operators, but there are other operators, like the $\mu-$functions of preferential logic which do not correspond to any such operator, and might even not preserve definability in the infinite case (see Definition 2.2.4 (page 42)). It seems more promising to order conditionals by the properties of their model choice functions, e.g., whether those functions are idempotent, etc., see Section 3.2.2 (page 101) .

Many conditionals can be based on binary relations, e.g. modal conditionals on accessibility relations, preferential consequence relations on preference relations, counterfactuals and theory revision on distance relations, etc. Thus, it is promising to look at those relations, and their properties to bring more order into the vast field of conditionals. D.Gabbay introduced reactive structures (see, e.g., [Gab04]), and added supplementary expressivity to structures based on binary relations, see [GS08b] and [GS08f]. In particular, it was shown there that we can have cumulativity without the basic properties of preferential structures (e.g., OR). This is discussed in Section 3.2.4 (page 102).

1.4.2 Additional structure on language and truth values

Normally, the language elements (propositional variables) are not structured. This is somewhat surprising, as, quite often, one variable will be more important than another. Size or weight might often be more important than colour for physical objects, etc. It is probably the mathematical tradition which was followed too closely. One of the authors gave a semantics to theory revision using a measure on language elements in [Sch91-1] and [Sch91-3], but, as far as we know, the subject was not treated in a larger context so far. The present book often works with independence of language elements, see in particular Chapter 4 (page 125) and Chapter 5 (page 165), and Hamming type relations and distances between models, where it need not be the case that all variables have the same weight. Thus, it is obvious to discuss this subject in the present text. It can also be fruitful to discuss sizes of subsets of the set of variables, so we may, e.g., neglect differences to classical logic if they concern only a “small” set of propositional variables.

On the other hand, classical truth values have a natural order, $FALSE < TRUE$, and we will sometimes work with more than 2 truth values, see in particular Chapter 4 (page 125) , but also Section 5.3.6 (page 203). So there is a natural question: do we also have a total order, or a boolean order, or another order on those sets of truth values? Or: Is there a distance between truth values, so that a change from value a to value b is smaller than a change from a to c?

There is a natural correspondence between semantical structures and truth values, which is best seen by an example: Take finite (intuitionistic) Goedel logics, see Section 4.4.3 (page 148), say, for simplicity with two worlds. Now, ϕ may hold nowhere, everywhere, or only in the second world (called “there”, in contrast to “here”, the first world). Thus, we can express the same situation by three truth values: 0 for nowhere, 1 for only “there”, 2 for everywhere.

In Section 3.3.6 (page 107), we will make some short remarks on “softening” concepts, like neglecting “small” fragments of a language, etc. This way, we can define, e.g., “soft” interpolation, where we need a small set of variables which are not in both formulas.

Inheritance systems, (see, e.g., [TH89], [THT86], [THT87], [TTH91], [Tou86], also [Sch93] and

[Sch97], [GS08e], [GS08f]), present many aspects of independence, (see Section 3.3.7 (page 107)). Thus, if two nodes are not connected by valid paths, they may have very different languages, as language elements have to be inherited, otherwise, they are undefined. In addition, a may inherit from b property c, but not property d, as we have a contradiction to d (or, even $\neg d$) via a different node b'. Theses are among the aspects which make them natural for common sense reasoning, but also quite different from traditional logics.

1.4.3 Representation for general revision, update, and counterfactuals

Revision (see [AGM85], and the discussion in Section 2.5 (page 90)), update (see [KM90]), and counterfactuals (see [Lew73] and [Sta68]) are special forms of conditionals, which received much interest in the artificial intelligence community. Explicitly or implicitly (see [LMS95], [LMS01]), they are based on a distance based semantics, working with "closest worlds". In the case of revision, we look at those worlds which are closest to the present *set* of worlds, in update and counterfactual, we look from each present world *individually* to the closest worlds, and then take the union. Obviously, the formal properties may be very different in the two cases.

There are two obvious generalizations possible, and sometimes necessary. First, "closest" worlds need not exist, there may be infinite descending chains of distances without minimal elements. Second, a distance or ranked order may force too many comparisons, when two distances or elements may just simply not be comparable. We address representation problems for these generalizations:

(1) We first generalize the notion of distance for revision semantics in Section 3.4.3 (page 112). We mostly consider symmetrical distances, so $d(a, b) = d(b, a)$, and we work with equivalence classes $[a, b]$. Unfortunately, one of the main tools in [LMS01], a loop condition, does not work any more, it is too close to rankedness.

We will have to work more in the spirit of general and smooth preferential structures to obtain representation. Unfortunately, revision does not allow many observations (see [LMS01], and, in particular, the impossibility results for revision ("Hamster Wheels") discussed in [Sch04]), so all we have (see Section 3.4.3.3 (page 114)) are results which use more conditions than what can be observed from revision observations. This problem is one of principles: we showed in [GS08a], see also [GS08f], that cumulativity suffices only to guarantee smoothness of the structure if the domain is closed under finite unions. But the union of two products need not be a product any more.

To solve the problem, we use a technique employed in [Sch96-1], using "witnesses" to testify for the conditions.

(2) We then discuss the limit version (when there are no minimal distances) for theory revision.

(3) In Section 3.4.4 (page 117), we turn to generalized update and counterfactuals. To solve this problem, we use a technique invented in [MS90], and adapt it to our situation. The basic idea is very simple: we begin (simplified) with some world x, and arrange the other worlds around x, as x sees them, by their relative distances. Suppose we consider now one those worlds, say y. Now we arrange the worlds around y, as y sees them. If we make all the new distances smaller than the old ones, we "cannot look back", etc. We continue this construction unboundedly (but finitely) often. If we are a little careful, everyone will only see what he is supposed to see. In a picture, we construct galaxies around a center, then

planets around suns, moons around planets, etc. The resulting construction is an $\mathcal{A}-$ranked structure, as discussed in [GS08d], see also [GS08f].

(4) In Section 3.4.5 (page 122), we discuss the corresponding syntactic conditions, using again ideas from [Sch96-1].

1.5 Interpolation

1.5.1 Introduction

The two chapters Chapter 4 (page 125) and Chapter 5 (page 165) are probably the core of the present book.

We treat very general interpolation problems for monotone and antitone, 2-valued and many-valued logics in Chapter 4 (page 125) , splitting the question in two parts, "semantic interpolation" and "syntactic interpolation", show that the first problem, existence of semantic interpolation, has a simple and general answer, and reduce the second question, existence of syntactic interpolation to a definability problem. For the latter, we examine the concrete example of finite Goedel logics. We can also show that the semantic problem has two "universal" solutions, which depend only on one formula and the shared variables.

In Chapter 5 (page 165), we investigate three variants of semantic interpolation for non-monotonic logics, in syntactic shorthand of the types $\phi \mid\sim \alpha \vdash \psi$, $\phi \vdash \alpha \mid\sim \psi$, and $\phi \mid\sim \alpha \mid\sim \psi$, where α is the interpolant, and see that two variants are closely related to multiplication laws about abstract size, defining (or originating from) the non-monotonic logics. The syntactic problem is analogous to that of the monotonic case.

1.5.1.1 Background

Interpolation for classical logic is well-known, see [Cra57], and there are also non-classical logics for which interpolation has been shown, e.g., for Circumscription, see [Ami02]. An extensive overview of interpolation is found in [GM05]. Chapter 1 of this book [GM05] gives a survey and a discussion and the chapter puts forward that interpolation can be viewed in many different ways and indeed 11 points of view of interpolation are discussed. The present text presents the semantic interpolation, this is a new 12th point of view.

1.5.2 Problem and Method

In classical logic, the problem of interpolation is to find for two formulas ϕ and ψ such that $\phi \vdash \psi$ a "simple" formula α such that $\phi \vdash \alpha \vdash \psi$. "Simple" is defined as: "expressed in the common language of ϕ and ψ".

Working on the semantic level has often advantages:

- results are robust under logically equivalent reformulations
- in many cases, the semantic level allows an easy reformulation as an algebraic problem, whose results can be generalized to other situations

- we can split a problem in two parts: a semantical problem, and the problem to find a syntactic counterpart (a definability problem)
- the semantics of many non-classical logics are built on relatively few basic notions like size, distance, etc., and we can thus make connections to other problems and logics
- in the case of preferential and similar logics, the very definition of the logic is already semantical (minimal models), so it is very natural to proceed on this level.

This strategy - translate to the semantic level, do the main work there, and then translate back - has proved fruitful also in the present case.

Looking back at the classical interpolation problem, and translating it to the semantic level, it becomes: Given $M(\phi) \subseteq M(\psi)$ (the models sets of ϕ and ψ), is there a "simple" model set A such that $M(\phi) \subseteq A \subseteq M(\psi)$? Or, more generally, given model sets $X \subseteq Y$, is there "simple" A such that $X \subseteq A \subseteq Y$? Of course, we have to define in a natural way, what "simple" means in our context. This is discussed below in Section 1.5.3.1 (page 20).

Our separation of the semantic from the syntactic question pays immediately:

(1) We see that monotonic (and antitonic) logics *always* have a semantical interpolant. But this interpolant may not be definable syntactically.

(2) More precisely, we see that there is a whole interval of interpolants in above situation.

(3) We also see that our analysis generalizes immediately to many valued logics, with the same result (existence of an interval of interpolants).

(4) Thus, the question remains: under what conditions does a syntactic interpolant exist?

(5) In non-monotonic logics, our analysis reveals a deep connection between semantic interpolation and questions about (abstract) multiplication of (abstract) size.

1.5.3 Monotone and antitone semantic and syntactic interpolation

We consider here the semantic property of monotony or antitony, in the following sense (in the two-valued case, the generalization to the many-valued case is straightforward):

Let $\vdash$ be some logic such that $\phi \vdash \psi$ implies $M(\phi) \subseteq M(\psi)$ (the monotone case) or $M(\psi) \subseteq M(\phi)$ (the antitone case).

In the many-valued case, the corresponding property is that $\rightarrow$ (or $\vdash$) respects $\leq$, the order on the truth values.

1.5.3.1 Semantic interpolation

The problem (for simplicity, for the 2-valued case) reads now:

If $M(\phi) \subseteq M(\psi)$ (or, symmetrically $M(\psi) \subseteq M(\phi)$), is there a "simple" model set A, such that $M(\phi) \subseteq A \subseteq M(\psi)$, or $M(\psi) \subseteq A \subseteq M(\phi)$. Obviously, the problem is the same in both cases. We will see that such A will always exist, so all such logics have semantic interpolation (but not necessarily also syntactic interpolation). We turn to the main conceptual problem, the definition of "simple".

The main conceptual problem is to define "simple model set". We have to look at the syntactic problem for guidance. Suppose ϕ is defined using propositional variables p and q, ψ using q and r. α has to be defined using only q. What are the models of α? By the very definition of validity in classical logic, neither p nor r have any influence on whether m is a model of α or not. Thus, if m is a model of α, we can modify m on p and r, and it will still be a model. Classical models are best seen as functions from the set of propositional variables to $\{true, false\}$, $\{t, f\}$, or so. In this terminology, any m with $m \models \alpha$ is "free" to choose the value for p and r, and we can write the model set A of α as $\{t, f\} \times M_q \times \{t, f\}$, where M_q is the set of values for q $\alpha-$models may have ($\emptyset$, $\{t\}$, $\{f\}$, $\{t, f\}$).

So, the semantic interpolation problem is to find sets which may be restricted on the common variables, but are simply the Cartesian product of the possible values for the other variables. To summarize: Let two model sets X and Y be given, where X itself is restricted on variables $\{p_1, \ldots, p_m\}$ (i.e. the Cartesian product for the rest), Y is restricted on $\{r_1, \ldots, r_n\}$, then we have to find a model set A which is restricted only on $\{p_1, \ldots, p_m\} \cap \{r_1, \ldots, r_n\}$, and such that $X \subseteq A \subseteq Y$, of course.

Formulated this way, our approach, the problem and its solution, has two trivial generalizations:

- for multi-valued logics we take the Cartesian product of more than just $\{t, f\}$.
- ϕ may be the hypothesis, and ψ the consequence, but also vice versa, there is no direction in the problem. Thus, any result for classical logic carries over to the core part of, e.g., preferential logics.

The main result for the situation with $X \subseteq Y$ is that there is always such a semantic interpolant A as described above (see Proposition 4.2.1 (page 131) for a simple case, and Proposition 4.2.3 (page 134) for the full picture). Our proof works also for "parallel interpolation", a concept introduced by Makinson et al., [KM07].

We explain and quote the latter result.

Suppose we have $f, g : M \to V$, where, intuitively, M is the set of all models, and V the set of all truth values. Thus, f and g give to each model a truth value, and, intuitively, f and g each code a model set, assigning to m TRUE iff m is in the model set, and FALSE iff not. We further assume that there is an order on the truth value set V. $\forall m \in M(f(m) \leq g(m))$ corresponds now to $M(\phi) \subseteq M(\psi)$, or $\phi \vdash \psi$ in classical logic. Each model m is itself a function from L, the set of propositional variables to V. Let now $J \subseteq L$. We say that f is insensitive to J iff the values of m on J are irrelevant: If $m \restriction (L-J) = m' \restriction (L-J)$, i.e., m and m' agree at least on all $p \in L-J$, then $f(m) = f(m')$. This corresponds to the situation where the variable p does not occur in the formula ϕ, then $M(\phi)$ is insensitive to p, as the value of any m on p does not matter for m being a model of ϕ, or not.

We need two more definitions:

Let $J' \subseteq L$, then $f^+(m_{J'}) := max\{f(m') : m' \restriction J' = m \restriction J'\}$ and $f^-(m_{J'}) := min\{f(m') : m' \restriction J' = m \restriction J'\}$.

We quote now Proposition 4.2.3 (page 134) :

Proposition 1.5.1

Let M be rich, $f, g : M \to V$, $f(m) \leq g(m)$ for all $m \in M$. Let $L = J \cup J' \cup J''$, let f be insensitive to J, g be insensitive to J''.

Then $f^+(m_{J'}) \leq g^-(m_{J'})$ for all $m_{J'} \in M \restriction J'$, and any $h : M \restriction J' \to V$ which is insensitive to $J \cup J''$ is an interpolant iff

$f^+(m_{J'}) \leq h(m_J m_{J'} m_{J''}) = h(m_{J'}) \leq g^-(m_{J'})$ for all $m_{J'} \in M \restriction J'$.

(h can be extended to the full M in a unique way, as it is insensitive to $J \cup J''$.)

See Diagram 4.2.1 (page 134).

1.5.3.2 The interval of interpolants

Our result has an additional reading: it defines an interval of interpolants, with lower bound $f^+(m_{J'})$ and upper bound $g^-(m_{J'})$. But these interpolants have a particular form. If they exist, i.e. iff $f \leq g$, then $f^+(m_{J'})$ depends only on f and J' (and m), but *not* on g, $g^-(m_{J'})$ only on g and J', *not* on f. Thus, they are universal, as we have to look only at one function and the set of common variables.

Moreover, we will see in Section 4.3.3 (page 137) that they correspond to simple operations on the normal forms in classical logic. This is not surprising, as we "simplify" potentially complicated model sets by replacing some coordinates with simple products. The question is, whether our logic allows to express this simplification, classical logic does.

1.5.3.3 Syntactic interpolation

Recall the problem described at the beginning of Section 1.5.3.1 (page 20). We were given $M(\phi) \subseteq M(\psi)$, and were looking for a "simple" model set A such that $M(\phi) \subseteq A \subseteq M(\psi)$. We just saw that such A exists, and were able to describe an interval of such A. But we have no guarantee that any such A is definable, i.e., that there is some α with $A = M(\alpha)$.

In classical logic, such α exists, see, e.g., Proposition 4.4.1 (page 147)), but also Section 4.3.3 (page 137). Basically, in classical logic, $f^+(m_{J'})$ and $g^-(m_{J'})$ correspond to simplifications of the formulas expressed in normal form, see Fact 4.3.3 (page 140) (in a different notation, which we will explain in a moment). This is not necessarily true in other logics, see Example 4.4.1 (page 157). (We find here again the importance of definability preservation, a concept introduced by one of us in [Sch92].)

If we have projections (simplifications), see Section 4.3 (page 136), we also have syntactic interpolation. At present, we do not know whether this is a necessary condition for all natural operators.

We can also turn the problem around, and just define suitable operators. This is done in Section 4.3.3 (page 137), Definition 4.3.2 (page 138) and Definition 4.3.3 (page 138) . There is a slight problem, as one of the operands is a *set* of propositional variables, and not a formula, as usual. One, but certainly not the only one, possibility is to take a formula (or the corresponding model set) and "extract" the "relevant" variables from it, i.e., those, which cannot be replaced by a product. Assume now that f is one of the generalized model "sets", then:

Given f, define

(1) $(f \uparrow J)(m) := sup\{f(m') : m' \in M,\, m \restriction J = m' \restriction J\}$

(2) $(f \downarrow J)(m) := inf\{f(m') : m' \in M,\, m \restriction J = m' \restriction J\}$

(3) $\phi!\psi$ by:
$f_{\phi!\psi} := f_\phi \uparrow (L - R(\psi))$

(4) $\phi?\psi$ by:
$f_{\phi?\psi} := f_\phi \downarrow (L - R(\psi))$

We then obtain for classical logic (see Fact 4.3.3 (page 140)):

Fact 1.5.2

Let $J := \{p_{1,1}, \ldots, p_{1,m_1}, \ldots, p_{n,1}, \ldots, p_{n,m_n}\}$

(1) Let $\phi_i := \pm p_{i,1} \wedge \ldots \wedge \pm p_{i,m_i}$ and $\psi_i := \pm q_{i,1} \wedge \ldots \wedge \pm q_{i,k_i}$, let $\phi := (\phi_1 \wedge \psi_1) \vee \ldots \vee (\phi_n \wedge \psi_n)$. Then $\phi \uparrow J = \phi_1 \vee \ldots \vee \phi_n$.

(2) Let $\phi_i := \pm p_{i,1} \vee \ldots \vee \pm p_{i,m_i}$ and $\psi_i := \pm q_{i,1} \vee \ldots \vee \pm q_{i,k_i}$, let $\phi := (\phi_1 \vee \psi_1) \wedge \ldots \wedge (\phi_n \vee \psi_n)$. Then $\phi \downarrow J = \phi_1 \wedge \ldots \wedge \phi_n$.

In a way, these operators are natural, as they simplify definable model sets, so they can be used as a criterion of the expressive strength of a language and logic: If X is definable, and Y is in some reasonable sense simpler than X, then Y should also be definable. If the language is not sufficiently strong, then we can introduce these operators, and have also syntactic interpolation.

1.5.3.4 Finite Goedel logics

The semantics of finite (intuitionistic) Goedel logics is a finite chain of worlds, which can also be expressed by a totally ordered set of truth values 0 ... n (see Section 4.4.3 (page 148)). Let FALSE and TRUE be the minimal and maximal truth values. ϕ has value false, iff it holds nowhere, and TRUE, iff it holds everywhere, it has value 1 iff it holds from world 2 onward, etc. The operators are classical $\wedge$ and $\vee$, negation $\neg$ is defined by $\neg(FALSE) = TRUE$ and $\neg(x) = FALSE$ otherwise. Implication $\rightarrow$ is defined by $\phi \rightarrow \psi$ is TRUE iff $\phi \leq \psi$ (as truth values), and the value of ψ otherwise.

More precisely, where f_ϕ is the model value function of the formula ϕ :

negation $\neg$ is defined by:

$$f_{\neg\phi}(m) := \begin{cases} TRUE \quad iff \quad f_\phi(m) = FALSE \\ \\ FALSE \qquad\quad otherwise \end{cases}$$

implication $\rightarrow$ is defined by:

$$f_{\phi\rightarrow\psi}(m) := \begin{cases} TRUE \quad iff \quad f_\phi(m) \leq f_\psi(m) \\ \\ f_\psi(m) \qquad\quad otherwise \end{cases}$$

see Definition 4.4.2 (page 148) in Section 4.4.3 (page 148). We show in Section 4.4.3.2 (page 157) the well-known result that such logics for 3 worlds (and thus 4 truth values) have no interpolation, whereas the corresponding logic for 2 worlds has interpolation. For the latter logic, we can still find a kind of normal form, though $\rightarrow$ cannot always be reduced. At least we can avoid nested implications, which is not possible in the former logic for 3 worlds.

We also discuss several "hand made" additional operators which allow us to define sufficiently many model sets to have syntactical interpolation - of course, we *know* that we have semantical interpolation. A more systematic approach was discussed above, the operators $\phi!\psi$ and $\phi?\psi$.

1.5.4 Laws about size and interpolation in non-monotonic logics

1.5.4.1 Various concepts of size and non-monotonic logics

A natural interpretation of the non-monotonic rule $\phi \mid\sim \psi$ is that the set of exceptional cases, i.e., those where ϕ holds, but not ψ, is a small subset of all the cases where ϕ holds, and the complement, i.e., the set of cases where ϕ and ψ hold, is a big subset of all $\phi-$cases.

This interpretation gives an abstract semantics to non-monotonic logic, in the sense that definitions and rules are translated to rules about model sets, without any structural justification of those rules, as they are given, e.g., by preferential structures, which provide structural semantics. Yet, they are extremely useful, as they allow us to concentrate on the essentials, forgetting about syntactical reformulations of semantically equivalent formulas, the laws derived from the standard proof theoretical rules incite to generalizations and modifications, and reveal deep connections but also differences. One of those insights is the connection between laws about size and (semantical) interpolation for non-monotonic logics, discussed in Chapter 5 (page 165) .

To put this abstract view a little more into perspective, we now present three alternative systems, also working with abstract size as a semantics for non-monotonic logics. (They will be repeated in the introduction of Chapter 5 (page 165).)

- the system of one of the authors for a first order setting, published in [Sch90] and elaborated in [Sch95-1],
- the system of S.Ben-David and R.Ben-Eliyahu, published in [BB94],
- the system of N.Friedman and J.Halpern, published in [FH96].

(1) Defaults as generalized quantifiers:

We first recall the definition of a "weak filter", made official in Definition 2.2.3 (page 42) :

Fix a base set X. A weak filter on or over X is a set $\mathcal{F} \subseteq \mathcal{P}(X)$, s.t. the following conditions hold:

(F1) $X \in \mathcal{F}$

(F2) $A \subseteq B \subseteq X$, $A \in \mathcal{F}$ imply $B \in \mathcal{F}$

$(F3')$ $A, B \in \mathcal{F}$ imply $A \cap B \neq \emptyset$.

We use weak filters on the semantical side, and add the following axioms on the syntactical side to a FOL axiomatisation:

1. $\nabla x\phi(x) \wedge \forall x(\phi(x) \to \psi(x)) \to \nabla x\psi(x)$,

2. $\nabla x\phi(x) \to \neg\nabla x\neg\phi(x)$,

3. $\forall x\phi(x) \to \nabla x\phi(x)$ and $\nabla x\phi(x) \to \exists x\phi(x)$.

A model is now a pair, consisting of a classical FOL model M, and a weak filter over its universe. Both sides are connected by the following definition, where $\mathcal{N}(M)$ is the weak filter on the universe of the classical model M :

$\langle M, \mathcal{N}(M)\rangle \models \nabla x\phi(x)$ iff there is $A \in \mathcal{N}(M)$ s.t. $\forall a \in A$ $(\langle M, \mathcal{N}(M)\rangle \models \phi[a])$.

Soundness and completeness is shown in [Sch95-1], see also [Sch04].

The extension to defaults with prerequisites by restricted quantifiers is straightforward.

(2) The system of S. Ben-David and R. Ben-Eliyahu:

Let $\mathcal{N}' := \{\mathcal{N}'(A) : A \subseteq U\}$ be a system of filters for $\mathcal{P}(U)$, i.e. each $\mathcal{N}'(A)$ is a filter over A. The conditions are (in slight modification):

UC': $B \in \mathcal{N}'(A) \to \mathcal{N}'(B) \subseteq \mathcal{N}'(A)$,

DC': $B \in \mathcal{N}'(A) \to \mathcal{N}'(A) \cap \mathcal{P}(B) \subseteq \mathcal{N}'(B)$,

RBC': $X \in \mathcal{N}'(A)$, $Y \in \mathcal{N}'(B) \to X \cup Y \in \mathcal{N}'(A \cup B)$,

SRM': $X \in \mathcal{N}'(A)$, $Y \subseteq A \to A - Y \in \mathcal{N}'(A) \vee X \cap Y \in \mathcal{N}'(Y)$,

GTS': $C \in \mathcal{N}'(A)$, $B \subseteq A \to C \cap B \in \mathcal{N}'(B)$.

(3) The system of N. Friedman and J. Halpern:

Let U be a set, $<$ a strict partial order on $\mathcal{P}(U)$, (i.e. $<$ is transitive, and contains no cycles). Consider the following conditions for $<$:

(B1) $A' \subseteq A < B \subseteq B' \to A' < B'$,

(B2) if A, B, C are pairwise disjoint, then $C < A \cup B$, $B < A \cup C \to B \cup C < A$,

(B3) $\emptyset < X$ for all $X \neq \emptyset$,

(B4) $A < B \to A < B-$A,

(B5) Let $X, Y \subseteq A$. If $A - X < X$, then $Y < A - Y$ or $Y - X < X \cap Y$.

The equivalence of the systems of [BB94] and [FH96] was shown in [Sch97-4], see also [Sch04].

Historical remarks: Our own view as abstract size was inspired by the classical filter approach, as used e.g. in mathematical measure theory. The first time that abstract size was related to nonmonotonic logics was, to our knowledge, in the second author's [Sch90] and [Sch95-1], and, independently, in [BB94]. The approach to size by partial orders is first discussed - to our knowledge - by N.Friedman and J.Halpern, see [FH96]. More detailed remarks can also be found in [GS08c], [GS09a], [GS08f]. A somewhat different approach is taken in [HM07].

Before we introduce the connection between interpolation and multiplicative laws about size, we give now some comments on the laws about size themselves.

1.5.4.2 Additive and multiplicative laws about size

We give here a short introduction to and some examples for additive and multiplicative laws about size. A detailed overview is presented in Table 5.1 (page 189), Table 5.2 (page 190), and Table 5.3 (page 191). (The first two tables have to be read together, they are too big to fit on one page.)

They show connections and how to develop a multitude of logical rules known from nonmonotonic logics by combining a small number of principles about size. We can use them as building blocks to construct the rules from. More precisely, "size" is to be read as "relative size", since it is essential to change the base sets.

In the first two tables, these principles are some basic and very natural postulates, (Opt), (iM), $(eM\mathcal{I})$, $(eM\mathcal{F})$, and a continuum of power of the notion of "small", or, dually, "big", from $(1 * s)$ to $(< \omega * s)$. From these, we can develop the rest except, essentially, Rational Monotony, and thus an infinity of different rules.

The probably easiest way to see a connection between non-monotonic logics and abstract size is by considering preferential structures. Preferential structures define principal filters, generated by the set of minimal elements, as follows: if $\phi \mid\sim \psi$ holds in such a structure, then $\mu(\phi) \subseteq M(\psi)$, where $\mu(\phi)$ is the set of minimal elements of $M(\phi)$. According to our ideas, we define a principal filter $\mathcal{F}$ over $M(\phi)$ by $X \in \mathcal{F}$ iff $\mu(\phi) \subseteq X \subseteq M(\phi)$. Thus, $M(\phi) \cap M(\neg\psi)$ will be a "small" subset of $M(\phi)$. (Recall that filters contain the "big" sets, and ideals the "small" sets.)

We can now go back and forth between rules on size and logical rules, e.g.:

(For details, see Table 5.1 (page 189), Table 5.2 (page 190), and Table 5.3 (page 191).)

(1) The "AND" rule corresponds to the filter property (finite intersections of big subsets are still big).

(2) "Right weakening" corresponds to the rule that supersets of big sets are still big.

(3) It is natural, but beyond filter properties themselves, to postulate that, if X is a small subset of Y, and $Y \subseteq Y'$, then X is also a small subset of Y'. We call such properties "coherence properties" between filters. This property corresponds to the logical rule (wOR).

(4) In the rule (CM_ω), usually called Cautious Monotony, we change the base set a little when going from $M(\alpha)$ to $M(\alpha \wedge \beta)$ (the change is small by the prerequisite $\alpha \mid\sim \beta$), and still have $\alpha \wedge \beta \mid\sim \beta'$, if we had $\alpha \mid\sim \beta'$. We see here a conceptually very different use of "small", as we now change the base set, over which the filter is defined, by a small amount.

(5) The rule of Rational Monotony is the last one in the first table, and somewhat isolated there. It is better to be seen as a multiplicative law, as described in the third table. It corresponds to the rule that the product of medium (i.e, neither big nor small) sets, still has medium size.

1.5.4.3 Interpolation and size

The connection between non-monotonic logic and the abstract concept of size was investigated in [GS09a], see also [GS08f]. There, we looked among other things at abstract addition of size. Here, we will show a connection to abstract multiplication of size. Our semantic approach used decomposition of set theoretical products. An important step was to write a set of models Σ as a product of some set Σ' (which was a restriction of Σ), and some full Cartesian product. So, when we speak about size, we will have (slightly simplified) some big subset Σ_1 of one product Π_1, and some big subset Σ_2 of another product Π_2, and will now check whether $\Sigma_1 \times \Sigma_2$ is a big subset of $\Pi_1 \times \Pi_2$. In shorthand, whether "$big * big = big$". (See Definition 5.2.1 (page 178) for precise definitions.) Such conditions are called coherence conditions, as they do not concern the notion of size itself, but the way the sizes defined for different base sets are connected. Our main results here are Proposition 5.3.3 (page 197) and Proposition 5.3.5 (page 198). They say that if the logic under investigation is defined from a notion of size which satisfies sufficiently many multiplicative conditions, then this logic will have interpolation of type three or even two, see Paragraph 1.5.4.3 (page 27).

Consider now some set product $X \times X'$. (Intuitively, X and X' are model sets on sublanguages J and J' of the whole language L.) When we have now a rule like: If Y is a big subset of X, and Y' a big subset of X', then $Y \times Y'$ is a big subset of $X \times X'$, and conversely, we can calculate consequences separately in the sublanguages, and put them together to have the overall consequences. But this is the principle behind interpolation: we can work with independent parts.

This is made precise in Definition 5.2.1 (page 178), in particular by the rule

$(\mu * 1) : \mu(X \times X') = \mu(X) \times \mu(X')$.

(Note that the conditions $(\mu * i)$ and $(\Sigma * i)$ are equivalent, as shown in Proposition 5.2.1 (page 179) (for principal filters).)

The main result is that the multiplicative size rule $(\mu * 1)$ entails non-monotonic interpolation of the form $\phi \mid\!\sim \alpha \mid\!\sim \psi$, see Proposition 5.3.5 (page 198).

We take now a closer look at interpolation for non-monotonic logic.

The three variants of interpolation

Consider preferential logic, a rule like $\phi \mid\!\sim \psi$. This means that $\mu(\phi) \subseteq M(\psi)$. So we go from $M(\phi)$ to $\mu(\phi)$, the minimal models of ϕ, and then to $M(\psi)$, and, abstractly, we have $M(\phi) \supseteq \mu(\phi) \subseteq M(\psi)$, so we have neither necessarily $M(\phi) \subseteq M(\psi)$, nor $M(\phi) \supseteq M(\psi)$, the relation between $M(\phi)$ and $M(\psi)$ may be more complicated. Thus, we have neither the monotone, nor the antitone case. For this reason, our general results for monotone or antitone logics do not hold any more.

But we also see here that classical logic is used, too. Suppose that there is ϕ' which describes exactly $\mu(\phi)$, then we can write $\phi \mid\!\sim \phi' \vdash \psi$.

So we can split preferential logic into a core part - going from ϕ to its minimal models - and a second part, which is just classical logic. (Similar decompositions are also natural for other non-monotonic logics.) Thus, preferential logic can be seen as a combination of two logics, the non-monotonic core, and classical logic. It is thus natural to consider variants of the interpolation problem, where $\mid\!\sim$ denotes again preferential logic, and $\vdash$ as usual classical logic:

Given $\phi \mid\!\sim \psi$, is there "simple" α such that

(1) $\phi \mid\!\sim \alpha \vdash \psi$, or

(2) $\phi \vdash \alpha \mid\!\sim \psi$, or

(3) $\phi \mid\!\sim \alpha \mid\!\sim \psi$?

In most cases, we will only consider the semantical version, as the problems of the syntactical version are very similar to those for monotonic logics. We turn to the variants.

(1) The first variant, $\phi \mid\sim \alpha \vdash \psi$, has a complete characterization in Proposition 5.3.2 (page 195), provided we have a suitable normal form (conjunctions of disjunctions). The condition says that the relevant variables of $\mu(\phi)$ have to be relevant for $M(\phi)$.

(2) The second variant, $\phi \vdash \alpha \mid\sim \psi$, is related to very (and in many cases, too) strong conditions about size. We do not have a complete characterization, only sufficient conditions about size. The size conditions we need are (see Definition 5.2.1 (page 178)):

the abovementioned $(\mu * 1)$, and,

$(\mu * 2) : \mu(X) \subseteq Y \Rightarrow \mu(X \upharpoonright A) \subseteq Y \upharpoonright A$

where X need not be a product any more.

The result is given in Proposition 5.3.3 (page 197).

Example 5.2.1 (page 181) shows that $(\mu * 2)$ seems too strong when compared to probability defined size. We repeat this example here, for the reader's convenience.

Example 1.5.1

Take a language of 5 propositional variables, with $X' := \{a, b, c\}$, $X'' := \{d, e\}$. Consider the model set $\Sigma := \{\pm a \pm b \pm cde, -a - b - c - d \pm e\}$, i.e. of 8 models of de and 2 models of $-d$. The models of de are 8/10 of all elements of Σ, so it is reasonable to call them a big subset of Σ. But its projection on X'' is only 1/3 of Σ''.

So we have a potential *decrease* when going to the coordinates.

This shows that weakening the prerequisite about X as done in $(\mu * 2)$ is not innocent.

We should, however, note that sufficiently modular preferential relations guarantee these very strong properties of the big sets, see Section 5.2.3 (page 181).

(3) We turn to the third variant, $\phi \mid\sim \alpha \mid\sim \psi$. This is probably the most interesting one, (a) as it is more general, it loosens the connection with classical logic, (b) it seems more natural as a rule, and (c) it is also connected to more natural laws about size. Again, we do not have a complete characterization, only sufficient conditions about size. Here, $(\mu * 1)$ suffices, and we have our main result about non-monotonic semanti interpolation, Proposition 5.3.5 (page 198), that $(\mu * 1)$ entails interpolation of the type $\phi \mid\sim \alpha \mid\sim \psi$.

Proposition 5.2.4 (page 182) shows that $(\mu * 1)$ is (roughly) equivalent to the relation properties

$(GH1)$ $\sigma \preceq \tau \wedge \sigma' \preceq \tau' \wedge (\sigma \prec \tau \vee \sigma' \prec \tau') \Rightarrow \sigma\sigma' \prec \tau\tau'$

(where $\sigma \preceq \tau$ iff $\sigma \prec \tau$ or $\sigma = \tau$)

$(GH2)$ $\sigma\sigma' \prec \tau\tau' \Rightarrow \sigma \prec \tau \vee \sigma' \prec \tau'$

of a preferential relation.

($(GH2)$ means that some compensation is possible, e.g., $\tau \prec \sigma$ might be the case, but $\sigma' \prec \tau'$ wins in the end, so $\sigma\sigma' \prec \tau\tau'$.)

There need not always be a semantical interpolation for the third variant, this is shown in Example 5.3.1 (page 193).

So we see that, roughly, semantic interpolation for nonmonotonic logics works when abstract size is defined in a modular way - and we find independence again. In a way, this is not surprising, as we use independent definition of validity for interpolation in classical logic, and we use independent definition of additional structure (relations or size) for interpolation in non-monotonic logic.

1.5.4.4 Hamming relations and size

As preferential relations are determined by a relation, and give rise to abstract notions of size and their manipulation, it is natural to take a close look at the corresponding properties of the relation. We already gave a few examples in the preceding sections, so we can be concise here. Our main definitions and results on this subject are to be found in Section 5.2.3 (page 181), where we also discuss distances with similar properties.

It is not surprising that we find various types of Hamming relations and distances in this context, as they are, by definition, modular. Neither is it surprising that we see them again in Chapter 6 (page 213), as we are interested there in independent ways to define neighbourhoods.

Basically, these relations and distances come in two flavours, the set and the counting variant. This is perhaps best illustrated by the Hamming distance of two sequence of finite, equal length. We can define the distance by the *set* of arguments where they differ, or by the *cardinality* of this set. The first results in possibly incomparable distances, the second allows "compensation", difference in one argument can be compensated by equality in another argument.

For definitions and results, also those connecting them to notions of size, see Section 5.2.3 (page 181) in particular Definition 5.2.2 (page 181). We then show in Proposition 5.2.4 (page 182) that (smooth) Hamming relations generate our size conditions when size is defined as above from a relation (the set of preferred elements generates the principal filter). Thus, Hamming relations determine logics which have interpolation, see Corollary 5.3.4 (page 198).

We define Hamming relations twice, in Section 5.2.3 (page 181), and in Section 6.3.1.3 (page 220), their uses and definitions differ slightly.

1.5.4.5 Equilibrium logic

Equilibrium logic, due to D.Pearce, A.Valverde, see [PV09] for motivation and further discussion, is based on the 3-valued finite Goedel logic, also called HT logic, HT for "here and there". Our results are presented in Section 5.3.6 (page 203).

Equilibrium logic (EQ) is defined by a choice function on the model set. First models have to be "total", no variable of the language may have 1 as value. Second, if $m \prec m'$, then m is considered better, and m' discarded, where $m \prec m'$ iff m and m' give value 0 to the same variables, and m gives value 2 to strictly less (as subset) variables than m' does.

We can define equilibrium logic by a preferential relation (taking care also of the first condition), but it is not smooth. Thus, our general results from the beginning of this section will not hold, and we have to work with "hand knitted" solutions. We first show that equilibrium logic has no interpolation of the form $\phi \vdash \alpha \mid\!\sim \psi$ or $\phi \mid\!\sim \alpha \vdash \psi$, then that is has interpolation of the form $\phi \mid\!\sim \alpha \mid\!\sim \psi$, and that the interpolant is also definable, i.e., equilibrium logic has semantic and syntactic interpolation of this form. Essentially, semantic interpolation is due to the fact that the preference relation is defined in a modular way, using individual variables - as always, when we have interpolation.

1.5.4.6 Interpolation for revision and argumentation

We have a short and simple result (Lemma 5.3.6 (page 203)) for interpolation in AGM revision. Unfortunately, we need the variables from both sides of the revision operator as can easily be seen by revising with TRUE. The reader is referred to Section 5.3.5 (page 203) for details.

Somewhat surprisingly, we also have an interpolation result for one form of argumentation, where we consider the set of arguments for a statement as the truth value of that statement. As we have maximum (set union), we have the lower bound used in Proposition 4.2.3 (page 134) for the monotonic case, and can show Fact 5.5.3 (page 211). See Section 5.5 (page 209) for details.

1.5.4.7 Language change to obtain products

To achieve interpolation and other results of independence, we often need to write a set of models as a non-trivial product. Sometimes, this is impossible, but an equivalent reformulation of the language can selve the problem.

As this might be interesting also for the non-specialists, we repeat Example 5.2.5 (page 192) here:

Example 1.5.2

Consider $p = 3$, and let

abc, $a\neg bc$, $a\neg b\neg c$, $\neg abc$, $\neg a\neg b\neg c$, $\neg ab\neg c$ be the $6 = 2 * 3$ positive cases,

$ab\neg c$, $\neg a\neg bc$ the negative ones. (It is coincidence that we can factorize positive and negative cases - probably iff one of the factors is the full product, here 2, it could also be 4 etc.)

We divide the cases by 3 new variables, grouping them together in positive and negative cases. a' is indifferent, we want this to be the independent factor, the negative ones will be put into $\neg b'\neg c'$. The procedure has to be made precise still. (n): negative

Let a' code the set abc, $a\neg bc$, $a\neg b\neg c$, $ab\neg c$ (n),

Let $\neg a'$ code $\neg a\neg bc$ (n), $\neg abc$, $\neg a\neg b\neg c$, $\neg ab\neg c$.

Let b' code abc, $a\neg bc$, $\neg a\neg b\neg c$, $\neg ab\neg c$

Let $\neg b'$ code $a\neg b\neg c$, $ab\neg c$ (n), $\neg a\neg bc$ (n), $\neg abc$

Let c' code abc, $a\neg b\neg c$, $\neg abc$, $\neg a\neg b\neg c$

Let $\neg c'$ code $a\neg bc$, $ab\neg c$ (n), $\neg a\neg bc$ (n), $\neg ab\neg c$

Then the 6 positive instances are

$\{a', \neg a'\} \times \{b'c', b'\neg c', \neg b'c'\}$, the negative ones

$\{a', \neg a'\} \times \{\neg b'\neg c'\}$

As we have 3 new variables, we code again all possible cases, so expressivity is the same.

Crucial here is that $6 = 3 * 2$, so we can just re-arrange the 6 models in a different way, see Fact 5.2.9 (page 192).

A similar result holds for the non-monotonic case, where the structure must be possible, we can then redefine the language.

All details are to be found in Section 5.2.5 (page 192).

1.5.5 Summary

We use our proven strategy of "divide et impera", transform the problem first in a semantical question, and then in a purely algebraic one:

- Classical and basic non-monotonic logic (looking for the sharpest consequence) have a surprising same answer, problems show up with definability when going back to the syntactical question.
- Thus, we separate algebraic from logical questions, and we see that there are logics with algebraic interpolation, but without logical interpolation, as the necessary sets of models are not definable in the language. This opens the way to making the language richer to obtain interpolation, when so desired.
- Full non-monotonic logic is more complicated, and finds a partial answer using the concept of size and a novel manipulation of it, justified by certain modular relations.
- Finally, our approach also has the advantage of short and elementary proofs.

1.6 Neighbourhood semantics

Neighbourhood semantics, see Chapter 6 (page 213), probably first introduced by D.Scott and R.Montague in [Sco70] and [Mon70], and already used for deontic logic by O.Pacheco in [Pac07], seem to be useful for many logics:

(1) in preferential logics, they describe the limit variant, where we consider neighbourhoods of an ideal, usually inexistent, situation,

(2) in approximative reasoning, they describe the approximations to the final result,

(3) in deontic and default logic, they describe the "good" situations, i.e., deontically acceptable, or where defaults have fired.

Neighbourhood semantics are used, when the "ideal" situation does not exist (e.g., preferential systems without minimal elements), or are too difficult to obtain (e.g., "perfect" deontic states).

1.6.1 Defining neighbourhoods

Neighbourhoods can be defined in various ways:

- by algebraic systems, like unions of intersections of certain sets (but not complements),

- quality relations, which say that some points are better than others, carrying over to sets of points,
- distance relations, which measure the distance to the perhaps inexistant ideal points.

The relations and distances may be given already by the underlying structure, e.g., in preferential structures, or they can be defined in a natural way, e.g., from a systems of sets, as in deontic logic or default logic. In these cases, we can define a distance between two points by the number or set of deontic requirements or default rules which one satisfies, but not the other. A quality relation is defined in a similar way: a point is better, if it satisfies more requirements or rules.

1.6.2 Additional requirements

With these tools, we can define properties neighbourhoods should have. E.g., we may require them to be downward closed, i.e., if $x \in N$, where N is a neighbourhood, $y \prec x$, y is better than x, then y should also be in N. This is a property we will certainly require in neighbourhood semantics for preferential structures (in the limit version). For these structures, we will also require that for every $x \notin N$, there should be some $y \in N$ with $y \prec x$. We may also require that, if $x \in N$, $y \notin N$, and y is in some aspect better than x, then there must be $z \in N$, which is better than both, so we have some kind of "ceteris paribus" improvement.

1.6.3 Connections between the various properties

There is a multitude of possible definitions (via distances, relations, set systems), and properties, so it is not surprising that one can investigate a multitude of connections between the different possible definitions of neighbourhoods. We cannot cover all possible connections, so we compare only a few cases, and the reader is invited to complete the picture for the cases which interest him. The connections we examined are presented in Section 6.3.4 (page 225).

1.6.4 Various uses of neighbourhood semantics

We also distinguish the different uses of the systems of sets thus characterized as neighbourhoods: we can look at all formulas which hold in (all or some) such sets (as in neighbourhood semantics for preferential logics), or at the formulas which exactly describe them. The latter reading avoids the infamous Ross paradox of deontic logic. This distinction is simple, but basic, and did probably not receive the attention it deserves, in the literature.

1.7 An abstract view on modularity and independence

1.7.1 Introduction

We see independence and modularity in many situations. Roughly, it means that we can use logic as a child who plays with building blocks and puts them together. The big picture is not more than the elements.

One example, which seems so natural, that it is hardly ever mentioned, is validity in classical propositonal logic. The validity of p in a model does not depend on the values a model assigns to other propositional variables. By induction, this property carries over to more complicated formulas. Consequently, the validity of a formula ϕ does not depend on the laguage: it suffices to know the values for the fragment in which ϕ is formulated to decide if ϕ holds or not. This is evident, but very important, it justifies what we call "semantic interpolation": Semantic interpolation will always hold for monotone or antitone logics. It does *not* follow that the language is sufficiently rich to describe such an interpolant, the latter will then be "syntactic interpolation". Syntactic interpolation can be guaranteed by the existence of suitable normal forms, which allow to treat model subsets independently.

For preferential non-monotonic logic, we see conditions for the resulting abstract notion of size and its multiplication which guarantee semantic interpolation also for those logics. Natural conditions for the preference relation result in such properties of abstract size.

Independence is also at the basis of an approach to theory revision due to Parikh and his co-authors, see [CP00]. Again, natural conditions on a distance relation result in such independent ways of revision.

The rule of Rational Monotony (see Table 2.2 (page 53)) can also be seen as independence: we can "cut up" the domain, and the same rules will still hold in the fragments.

1.7.2 Abstract definition of independence

Diagram 1.7.1

$$
\begin{tikzcd}
\Sigma_1 \arrow[rr] \arrow[dr] & & f(\Sigma_1) \arrow[dr, leftrightarrow] & \\
 & \Sigma_1 \circ \Sigma_2 \arrow[rr] & & f(\Sigma_1 \circ \Sigma_2) = f(\Sigma_1) \circ' f(\Sigma_2) \\
\Sigma_2 \arrow[rr] \arrow[ur] & & f(\Sigma_2) \arrow[ur, leftrightarrow] &
\end{tikzcd}
$$

Note that $\circ$ and $\circ'$ might be different

Independence

The right notion of independence in our context seems to be:

We have compositions $\circ$ and $\circ'$, and an operation f. We can calculate $f(\Sigma_1 \circ \Sigma_2)$ from $f(\Sigma_1)$ and $f(\Sigma_2)$, but also conversely, given $f(\Sigma_1 \circ \Sigma_2)$ we can calculate $f(\Sigma_1)$ and $f(\Sigma_2)$. Of course, in other contexts, other notions of independence might be adequate. More precisely:

Definition 1.7.1

Let $f : \mathcal{D} \to \mathcal{C}$ be any function from domain $\mathcal{D}$ to co-domain $\mathcal{C}$. Let $\circ$ be a "composition function" $\circ : \mathcal{D} \times \mathcal{D} \to \mathcal{D}$, likewise for $\circ' : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$.

We say that $\langle f, \circ, \circ' \rangle$ are independent iff for any $\Sigma_i \in \mathcal{D}$

(1) $f(\Sigma_1 \circ \Sigma_2) = f(\Sigma_1) \circ' f(\Sigma_2)$,

(2) we can recover $f(\Sigma_i)$ from $f(\Sigma_1 \circ \Sigma_2)$, provided we know how $\Sigma_1 \circ \Sigma_2$ splits into the Σ_i, without using f again.

1.7.2.1 Discussion

(1) Ranked structures satisfy it:

Let $\circ = \circ' = \cup$. Let f be the minimal model operator μ of preferential logic. Let $X, Y \subseteq X \cup Y$ have (at least) medium size, i.e. $X \cap \mu(X \cup Y) \neq \emptyset$, $Y \cap \mu(X \cup Y) \neq \emptyset$, (see below, Section 5.2.1.1 (page 171)). Then $\mu(X \cup Y) = \mu(X) \cup \mu(Y)$, and $\mu(X) = \mu(X \cup Y) \cap X$, $\mu(Y) = \mu(X \cup Y) \cap Y$.

(2) Consistent classical formulas and their interpretation satisfy it:

Let $\circ$ be conjunction in the composed language, $\circ'$ be model set intersection, $f(\phi) = M(\phi)$. Let ϕ, ψ be classical formulas, defined on disjoint language fragments $\mathcal{L}$, $\mathcal{L}'$ of some language $\mathcal{L}''$. Then $f(\phi \wedge \psi) = M(\phi) \cap M(\psi)$, and $M(\phi)$ is the projection of $M(\phi) \cap M(\psi)$ onto the (models of) language $\mathcal{L}$, likewise for $M(\psi)$. This is due to the way validity is defined, using only variables which occur in the formula.

As a consequence, monotonic logic has semantical interpolation - see [GS09c], and below, Section 5.3.6.1 (page 203). The definition of being insensitive is justified by this modularity.

(3) It does not hold for inconsistent classical formulas: We cannot recover $M(a \wedge \neg a)$ and $M(b)$ from $M(a \wedge \neg a \wedge b)$, as we do not know where the inconsistency came from. The basic reason is trivial: One empty factor suffices to make the whole product empty, and we do not know which factor was the culprit. See Section 1.7.3.3 (page 36) for the discussion of a remedy.

(4) Preferential logic satisfies it under certain conditions:

If $\mu(X \times Y) = \mu(X) \times \mu(Y)$ holds for model products and $\mid\!\sim$, then it holds by definition. An important consequence is that such a logic has interpolation of the form $\mid\!\sim \circ \mid\!\sim$, see Section 5.3.4 (page 198).

(5) Modular revision a la Parikh, see [CP00], is based on a similar idea.

1.7.3 Other aspects of independence

1.7.3.1 Existence of normal forms

We may see the existence of conjunctive and disjunctive normal forms as a form of independence: A formula may be split into elementary parts, which are then put together by the standard operations of inf ($\wedge$) and sup ($\vee$), resulting immediately in the existence of syntactic interpolation, as both the upper and lower limits of interpolation are definable. Note that higher finite Goedel logics do not allow these operations, basically as we cannot always decompose nested intuitionistic implication.

1.7.3.2 Language change

Independence of language fragments gives us the following perspectives:

(1) it makes independent and parallel treatment of fragments possible, and offers thus efficient treatment in applications (descriptive logics etc.):

Consider $X = X' \cup X''$, where X', X'' are disjoint. Suppose size is calculated independently, in the following sense: Let $Y \subseteq X$, then $Z \subseteq Y$ is big iff $Z \cap X' \subseteq Y \cap X'$ and $Z \cap X'' \subseteq Y \cap X''$ both are big. We can then calculate size independently.

(2) it results in new rules similar to the classical ones like AND, OR, Cumulativity, etc. We can thus obtain postulates about reasonable behaviour, but also classification by those rules, see Table 5.3 (page 191), Scenario 2, Logical property.

(3) it sheds light on notions like "ceteris paribus", which we saw in the context of obligations, see [GS08g], and Definition 6.3.11 (page 224),

(4) it clarifies notions like "normal with respect to ϕ, but not ψ", see [GS08e] and [GS08f],

(5) it helps to understand e.g. inheritance diagrams where arrows make other information accessible, and we need an underlying mechanism to combine bits of information, given in different languages, see again [GS08e] and [GS08f].

1.7.3.3 A relevance problem

Consider the formula $\phi := a \wedge \neg a \wedge b$. Then $M(\phi) = \emptyset$. But we cannot recover where the problem came from (it might also come from $b \wedge \neg b$), and this results in the EFQ rule. We now discuss one, purely algebraic, approach to remedy.

Consider 3 valued models, with a new value b for both, in addition to t and f. Above formula would then have the model $m(a) = b$, $m(b) = t$. So there is a model, EFQ fails, and we can recover the culprit.

To have the usual behaviour of $\wedge$ as intersection, it might be good to change the definition so that $m(x) = b$ is always a model. Then $M(b) = \{m(b) = t, m'(b) = b\}$, $M(\neg b) = \{m(b) = f, m'(b) = b\}$, and $M(b \wedge \neg b) = \{m'(b) = b\}$.

It is not yet clear which version to choose, and we have no syntactic characterization.

1.7.3.4 Small subspaces

When considering small subsets in nonmonotonic logic, we neglect small subsets of models. What is the analogue when considering small subspaces, i.e. when $J = J' \cup J''$, with J'' small in J in the sense of nonmonotonic logic?

It is perhaps easiest to consider the relation based approach first. So we have an order on $\Pi J'$ and one on $\Pi J''$, J'' is small, and we want to know how to construct a corresponding order on ΠJ. Two solutions come to mind:

- a less radical one: we make a lexicographic ordering, where the one on $\Pi J'$ has precedence over the one on $\Pi J''$,
- a more radical one: we totally forget about the ordering of $\Pi J''$, i.e. we do as if the ordering on $\Pi J''$ were the empty set, i.e. $\sigma'\sigma'' \prec \tau'\tau''$ iff $\sigma' \prec \tau'$ and $\sigma'' = \tau''$.

  We call this condition $forget(J'')$.

The less radical one is already covered by our relation conditions (GH) see Definition 5.2.2 (page 181). The more radical one is probably more interesting. Suppose ϕ' is written in language J', ϕ'' in language J'', we then have

$\phi' \wedge \phi'' \mid\sim \psi' \wedge \psi''$ iff $\phi' \mid\sim \psi'$ and $\phi'' \vdash \psi''$.

This approach is of course the same as considering on the small coordinate only ALL as a big subset, (see the lines $x * 1/1 * x$ in Table 5.3 (page 191)).

1.8 Conclusion and outlook

In Section 7.2 (page 236), we argue that logics which diverge from classical logic in the sense that they allow to conclude more or less than classical logic concludes need an additional fundamental concept, a $justification$. Classical logic has language and truth values, proof theory, and semantics. Here, we need more, justification, why we are allowed to conclude more or less. We have to show that the price we pay (divergence from truth) is justified, e.g., by more efficient reasoning, conjectures which "pay", etc.

We think that we need a new fundamental concept, which is on the same level as proof theory and semantics.

This is an open research problem, but it seems that our tools like abstract manipulation of abstract size are sufficient to attack it.

1.9 Previously published material, acknowledgements

This text builds upon previous research by the authors. To make the text self-contained, it is therefore necessary to repeat some previously published material. We give now the parts concerned and their sources.

All parts of Chapter 2 (page 39) which are not marked as new material were published in some or all of [Sch04], [GS08b], [GS08c], [GS09a], [GS08f].

The additive laws on abstract size (see Section 5.2.1 (page 171)) were published in [GS09a] and [GS08f].

The formal material of Chapter 6 (page 213) was already published in [GS08f], it is put here in a wider perspective.

Finally, we would like to thank D.Makinson and D.Pearce for useful comments and very interesting questions.

Chapter 2

Basic definitions

2.1 Introduction

2.1.1 Overview of this chapter

This chapter contains basic definitions and results, sometimes slightly beyond the immediate need of this book, as we also want to put our work a bit more in perspective, and make it self-contained, for the convenience of the reader. Most of the material of this chapter (unless marked as "new") was published previously, see [Sch04], [GS08b], [GS08c], [GS09a], and [GS08f].

We begin with basic algebraic and logical definitions, including in particular many laws of non-monotonic logics, in their syntactic and semantic variants, showing also the connections between both sides, see Definition 2.2.6 (page 44) and the tables Table 2.1 (page 52) and Table 2.2 (page 53).

It seems to be a little known result that even the classical operators permit an unusual interpretation in the infinite case, but we claim no originality, see Example 2.2.1 (page 44).

We would like to emphasize the importance of the definability preservation (dp) property. In the infinite case, not all model sets X are definable, i.e., there is some formula ϕ or theory T such that $X = M(\phi)$ - the models of ϕ - or $X = M(T)$ - the models of T. It is by no means evident that a model choice function μ, applied to a definable model set, gives us back again a definable model set (is definability preserving, or dp). If μ does not have this property, some representation results will not hold, which hold if μ is dp, and representation results become much more complicated, see [Sch04] for positive and for impossibility results. In our present context, definability is again an important concept. Even if we have semantic interpolation, if language and operators are not strong enough, we cannot define the semantic interpolants, so we have semantic, but not syntactic interpolation. Examples are found in finite Goedel logics, see Section 4.4 (page 146). New operators guaranteeing the definability of particularly interesting, "universal" interpolants, see Definition 4.3.1 (page 136), are discussed in Section 4.3 (page 136) . They are intricately related to the existence of conjunctive and disjunctive normal forms, as discussed in Section 4.3.3 (page 137).

We conclude this part with a - to our knowledge - unpublished result that we can define only countably many inconsistent formulas, see Example 2.2.2 (page 46). (The question is due to D.Makinson.)

We then give a detailed introduction into the basic concepts of many-valued logics, again, as readers might not be so familiar with the generalizations from 2-valued to many-valued logic. In particular, the nice correspondence between 2-valued functions and sets does not hold any more, so we have to work with arbitrary functions. We have to re-define what a definable model "set" is, and what semantical interpolation means for many-valued logic. Table 2.3 (page 54) gives an overview.

We then give an introduction to preferential structures and the logic they define. These structures are among the best examined semantics for non-monotonic logics, and Chapter 5 (page 165) is also based on the investigation of such structures. We first introduce the minimal variant, and then the limit variant. The first variant is the ususal one, the second is needed to deal with cases where there are no minimal models, due to infinite descending chains. (The first variant was introduced by Y.Shoham in [Sho87b], the second variant by P.Siegel et al. in [BS85]. It should, however, be emphasized, that preferential models were introduced as a semantics for deontic logic long before they were investigated as a semantics for non-monotonic logic, see [Han69]). The limit variant was further investigated in [Sch04], and we refer the reader there for representation and impossibility results. An overview of representation results for the minimal variant is given in Table 2.4 (page 59).

We introduce a new concept in this section on preferential structures, "bubble structures", which, we think, present a useful tool for abstraction, and are a semantic variant of independence in preferential structures. Here, we have a global preferential structure between subsets ("bubbles") of the model set, and a fine scale structure inside those subsets. Seen from the outside, all elements of a bubble behave the same way, so the whole set can be treated as one element, on the inside, we see a finer structure.

Moreover, new material on many-valued preferential structures is included.

We then go into details in the section on IBRS, introduced by D.Gabbay, see [Gab04], and further investigated in [GS08b] and [GS08f], as they are not so much common knowledge. We also discuss here if and how the limit version of preferential structures might be applied to reactive structures.

We then present theory revision, as introduced by Alchorron, Gardenfors, and Makinson, see [AGM85]. Again, we also build on previous results by (here, one of) the authors, when we discuss distance based revision, introduced by Lehmann, Magidor, and Schlechta, see [LMS95], [LMS01], and elaborated in [Sch04]. We also include a short paragraph on new material for theory revision based on many-valued logic.

2.2 Basic algebraic and logical definitions

Notation 2.2.1

We use sometimes FOL as abbreviation for first order logic, and NML for nonmonotonic logic. To avoid Latex complications in bigger expressions, we replace $\widetilde{xxxxx}$ by $\overbrace{xxxxx}$.

Definition 2.2.1

(1) We use $\mathcal{P}$ to denote the power set operator.

$\Pi\{X_i : i \in I\} := \{g : g : I \to \bigcup\{X_i : i \in I\}, \forall i \in I.g(i) \in X_i\}$ is the general Cartesian product, $X \times X'$ is the binary Cartesian product.

$card(X)$ shall denote the cardinality of X, and V the set-theoretic universe we work in - the class of all sets.

Given a set of pairs $\mathcal{X}$, and a set X, we denote by $\mathcal{X} \upharpoonright X := \{\langle x, i\rangle \in \mathcal{X} : x \in X\}$. (When the context is clear, we will sometime simply write X for $\mathcal{X} \upharpoonright X$.)

We will use the same notation $\upharpoonright$ to denote the restriction of functions and in particular of sequences to a subset of the domain.

If Σ is a set of sequences over an index set X, and $X' \subseteq X$, we will abuse notation and also write $\Sigma \upharpoonright X'$ for $\{\sigma \upharpoonright X' : \sigma \in \Sigma\}$.

Concatenation of sequences, e.g., of σ and σ', will be denoted by juxtaposition: $\sigma\sigma'$.

(2) $A \subseteq B$ will denote that A is a subset of B or equal to B, and $A \subset B$ that A is a proper subset of B, likewise for $A \supseteq B$ and $A \supset B$.

Given some fixed set U we work in, and $X \subseteq U$, then $\boldsymbol{C}(X) := U - X$.

(3) If $\mathcal{Y} \subseteq \mathcal{P}(X)$ for some X, we say that $\mathcal{Y}$ satisfies

($\cap$) iff it is closed under finite intersections,

($\bigcap$) iff it is closed under arbitrary intersections,

($\cup$) iff it is closed under finite unions,

($\bigcup$) iff it is closed under arbitrary unions,

($\boldsymbol{C}$) iff it is closed under complementation,

($-$) iff it is closed under set difference.

(4) We will sometimes write $A = B \parallel C$ for: $A = B$, or $A = C$, or $A = B \cup C$.

We make ample and tacit use of the Axiom of Choice.

Definition 2.2.2

$\prec^*$ will denote the transitive closure of the relation $\prec$. If a relation $<$, $\prec$, or similar is given, $a \perp b$ will express that a and b are $< -$ (or $\prec -$) incomparable - context will tell. Given any relation $<$, $\leq$ will stand for $<$ or $=$, conversely, given $\leq$, $<$ will stand for $\leq$, but not $=$, similarly for $\prec$ etc.

Definition 2.2.3

Fix a base set X.

A (weak) filter on or over X is a set $\mathcal{F} \subseteq \mathcal{P}(X)$, s.t. $(F1)-(F3)$ (($F1$), (F2), ($F3'$) respectively) hold:

(F1) $X \in \mathcal{F}$

(F2) $A \subseteq B \subseteq X$, $A \in \mathcal{F}$ imply $B \in \mathcal{F}$

(F3) $A, B \in \mathcal{F}$ imply $A \cap B \in \mathcal{F}$

($F3'$) $A, B \in \mathcal{F}$ imply $A \cap B \neq \emptyset$.

So a weak filter satisfies ($F3'$) instead of (F3).

A filter is called a principal filter iff there is $X' \subseteq X$ s.t. $\mathcal{F} = \{A : X' \subseteq A \subseteq X\}$.

An (weak) ideal on or over X is a set $\mathcal{I} \subseteq \mathcal{P}(X)$, s.t. $(I1)-(I3)$ (($I1$), (I2), ($I3'$) respectively) hold: (I1) $\emptyset \in \mathcal{I}$

(I2) $A \subseteq B \subseteq X$, $B \in \mathcal{I}$ imply $A \in \mathcal{I}$

(I3) $A, B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$

($I3'$) $A, B \in \mathcal{I}$ imply $A \cup B \neq X$.

So a weak ideal satisfies ($I3'$) instead of (I3).

A filter is an abstract notion of size, elements of a filter on X are called big subsets of X, their complements are called small, and the rest have medium size. The dual applies to ideals, this is justified by the following trivial fact:

Fact 2.2.1

If $\mathcal{F}$ is a (weak) filter on X, then $\mathcal{I} := \{X - A : A \in \mathcal{F}\}$ is a (weak) ideal on X, if $\mathcal{I}$ is a (weak) ideal on X, then $\mathcal{F} := \{X - A : A \in \mathcal{F}\}$ is a (weak) filter on X.

Definition 2.2.4

(1) V will be the set of truth values when there are more than the classical ones, TRUE and FALSE.

We work here in a classical propositional language $\mathcal{L}$, a theory T will be an arbitrary set of formulas. Formulas will often be named ϕ, ψ, etc., theories T, S, etc.

$v(\mathcal{L})$ or simply L will be the set of propositional variables of $\mathcal{L}$.

$F(\mathcal{L})$ will be the set of formulas of $\mathcal{L}$.

A propositional model m will be a function from the set of propositional variables to the set of truth values - V when we have more than two truth values.

$M_{\mathcal{L}}$ or simply M when the context is clear, will be the set of (classical) models for $\mathcal{L}$, $M(T)$ or M_T is the set of models of T, likewise $M(\phi)$ for a formula ϕ.

(2) $\boldsymbol{D}_{\mathcal{L}} := \{M(T) : T$ a theory in $\mathcal{L}\}$, the set of *definable* model sets.

Note that, in classical propositional logic, $\emptyset, M_{\mathcal{L}} \in \boldsymbol{D}_{\mathcal{L}}$, $\boldsymbol{D}_{\mathcal{L}}$ contains singletons, is closed under arbitrary intersections and finite unions.

An operation $f : \mathcal{Y} \to \mathcal{P}(M_{\mathcal{L}})$ for $\mathcal{Y} \subseteq \mathcal{P}(M_{\mathcal{L}})$ is called *definability preserving* , (*dp*) or (μdp) in short, iff for all $X \in \boldsymbol{D}_{\mathcal{L}} \cap \mathcal{Y}$ $f(X) \in \boldsymbol{D}_{\mathcal{L}}$.

We will also use (μdp) for binary functions $f : \mathcal{Y} \times \mathcal{Y} \to \mathcal{P}(M_{\mathcal{L}})$ - as needed for theory revision - with the obvious meaning.

(3) $\vdash$ will be classical derivability, and

$\overline{T} := \{\phi : T \vdash \phi\}$, the closure of T under $\vdash$.

(4) $Con(.)$ will stand for classical consistency, so $Con(\phi)$ will mean that ϕ is classical consistent, likewise for $Con(T)$. $Con(T, T')$ will stand for $Con(T \cup T')$, etc.

(5) Given a consequence relation $\mid\!\sim$, we define

$\overline{\overline{T}} := \{\phi : T \mid\!\sim \phi\}$.

(There is no fear of confusion with $\overline{T}$, as it just is not useful to close twice under classical logic.)

(6) $T \vee T' := \{\phi \vee \phi' : \phi \in T, \phi' \in T'\}$.

(7) If $X \subseteq M_{\mathcal{L}}$, then $Th(X) := \{\phi : X \models \phi\}$, likewise for $Th(m)$, $m \in M_{\mathcal{L}}$. ($\models$ will usually be classical validity.)

In the following, the X_i are arbitrary (non-empty) sets, standing for the set of truth values, and J is intuitively the set of propositional variables. So any element of $\Pi\{X_i : i \in J\}$ stands for a propositional model. Inessential variables in a set Σ are those which do not have any influence on the truth of the formula whose model set is Σ.

Definition 2.2.5

Let $\Sigma \subseteq \Pi := \Pi\{X_i : i \in J\}$. Define:

(1) For $\sigma, \sigma' \in \Sigma$, $J' \subseteq J$, define:

$\sigma \sim_{J'} \sigma' :\Leftrightarrow \forall x \in J' \sigma(x) = \sigma'(x)$.

(2) $I(\Sigma) := \{i \in J : \Sigma = \Sigma \restriction (J - \{i\}) \times X_i\}$ (up to re-ordering), (the irrelevant or inessential i) and

$R(\Sigma) := J - I(\Sigma)$ (the relevant or essential i).

Fact 2.2.2

(1) $\Sigma = \Sigma \restriction R(\Sigma) \times \Pi \restriction I(\Sigma)$ (up to re-ordering)

(2) $\sigma \restriction R(\Sigma) = \sigma' \restriction R(\Sigma) \wedge \sigma \in \Sigma \Rightarrow \sigma' \in \Sigma$.

Proof

(1) Enumerate $I(\Sigma)$, $I(\Sigma) = \{i : i < \kappa\}$. Define $\Sigma_j := \Sigma \restriction (R(\Sigma) \cup \{i : i < j\})$. We show by induction that $\Sigma_j = \Sigma \restriction R(\Sigma) \times \Pi \restriction (I(\Sigma) \cap j)$ for $j \leq \kappa$ (up to re-ordering).

$j = 0$ is trivial - there is nothing to show.

$j \to j+1$: This follows from the induction hypothesis and the definition of $I(\Sigma)$.

j is a limit ordinal: Any sequence of length j can be written as the coherent union of shorter sequences, and these are in both sets, as the result holds for $j' < j$ by induction hypothesis.

(2) Trivial.

□

We give the following example which shows that even the operators of classical propositional logic may be interpreted in a non-standard way (in the infinite case). We give this example, as the basic argument is about definability, and thus fully in our context.

Example 2.2.1

Consider a (countable for simplicity) infinite propositional language. Consider the usual model set M, and take any countable set of models away, resulting in M'. Interpret the variables as usual, p by $M(p)$, and let $M'(p) := M(p) \cap M'$. Interpret $\neg$ and $\wedge$ as usual, i.e., complement and intersection, but now in M', define $\vee$, $\to$ from $\neg$ and $\wedge$. Then $M'(\neg\phi) := M' - M'(\phi) = M' - (M' \cap M(\phi)) = M' \cap M(\neg\phi)$, $M'(\phi \wedge \psi) = M'(\phi) \cap M'(\psi) = M' \cap M(\phi) \cap M(\psi)$ by induction, so we have $M'(\phi) = M' \cap M(\phi)$. Consequently, the usual axioms hold, and Modus Ponens is a valid rule. For completeness, we have to show that every consistent formula has a model. Let ϕ be consistent, then $M'(\phi) = M' \cap M(\phi)$, but as ϕ is consistent, it has uncountably many models, so it also has a model in M'. □

To put our work more into perspective, we repeat now material from [GS08c]. This gives the main definitions and rules for non-monotonic logics, see Table 2.1 (page 52) and Table 2.2 (page 53), "Logical rules, definitions and connections".

Definition 2.2.6

The definitions are given in Table 2.1 (page 52), "Logical rules, definitions and connections Part I", and Table 2.2 (page 53), "Logical rules, definitions and connections Part II", which also show connections between different versions of rules, the semantics, and rules about size. (The tables are split in two, as they would not fit onto one page otherwise.)

Explanation of the tables:

(1) The first table gives the basic properties, the second table those for Cumulativity and Rational Monotony.

(2) The difference between the first two columns is that the first column treats the formula version of the rule, the second the more general theory (i.e., set of formulas) version.

(3) "Corr." stands for "Correspondence".

(4) The third column, "Corr.", is to be understood as follows:

Let a logic $\mid\sim$ satisfy (LLE) and (CCL), and define a function $f : \boldsymbol{D}_{\mathcal{L}} \to \boldsymbol{D}_{\mathcal{L}}$ by $f(M(T)) := M(\overline{\overline{T}})$. Then f is well defined, satisfies (μdp), and $\overline{\overline{T}} = Th(f(M(T)))$.

If $\mid\sim$ satisfies a rule in the left hand side, then - provided the additional properties noted in the middle for $\Rightarrow$ hold, too - f will satisfy the property in the right hand side.

Conversely:

If $f : \mathcal{Y} \to \mathcal{P}(M_{\mathcal{L}})$ is a function, with $\boldsymbol{D}_{\mathcal{L}} \subseteq \mathcal{Y}$, and we define a logic $\mid\sim$ by $\overline{\overline{T}} := Th(f(M(T)))$, then $\mid\sim$ satisfies (LLE) and (CCL). If f satisfies (μdp), then $f(M(T)) = M(\overline{\overline{T}})$.

If f satisfies a property in the right hand side, then - provided the additional properties noted in the middle for $\Leftarrow$ hold, too - $\mid\sim$ will satisfy the property in the left hand side.

(5) We use the following abbreviations for those supplementary conditions in the "Correspondence" columns:

"$T = \phi$" means that, if one of the theories (the one named the same way in Definition 2.2.6 (page 44)) is equivalent to a formula, we do not need (μdp).

$-(\mu dp)$ stands for "without (μdp)".

(6) $A = B \parallel C$ will abbreviate $A = B$, or $A = C$, or $A = B \cup C$.

Further comments:

(1) (PR) is also called $infinite\ conditionalization$ We choose this name for its central role for preferential structures (PR) or (μPR).

(2) The system of rules (AND) (OR) (LLE) (RW) (SC) (CP) (CM) (CUM) is also called system P (for preferential). Adding $(RatM)$ gives the system R (for rationality or rankedness).

Roughly: Smooth preferential structures generate logics satisfying system P, while ranked structures generate logics satisfying system R.

(3) A logic satisfying (REF), $(ResM)$, and (CUT) is called a *consequence relation*.

(4) (LLE) and(CCL) will hold automatically, whenever we work with model sets.

(5) (AND) is obviously closely related to filters, and corresponds to closure under finite intersections. (RW) corresponds to upward closure of filters.

More precisely, validity of both depend on the definition, and the direction we consider.

Given f and $(\mu \subseteq)$, $f(X) \subseteq X$ generates a principal filter: $\{X' \subseteq X : f(X) \subseteq X'\}$, with the definition: If $X = M(T)$, then $T \mid\sim \phi$ iff $f(X) \subseteq M(\phi)$. Validity of (AND) and (RW) are then trivial.

Conversely, we can define for $X = M(T)$

$\mathcal{X} := \{X' \subseteq X : \exists\phi(X' = X \cap M(\phi) \text{ and } T \mid\!\sim \phi)\}$.

(AND) then makes $\mathcal{X}$ closed under finite intersections, and (RW) makes $\mathcal{X}$ upward closed. This is in the infinite case usually not yet a filter, as not all subsets of X need to be definable this way. In this case, we complete $\mathcal{X}$ by adding all X'' such that there is $X' \subseteq X'' \subseteq X$, $X' \in \mathcal{X}$.

Alternatively, we can define

$\mathcal{X} := \{X' \subseteq X : \bigcap\{X \cap M(\phi) : T \mid\!\sim \phi\} \subseteq X'\}$.

(6) (SC) corresponds to the choice of a subset.

(7) (CP) is somewhat delicate, as it presupposes that the chosen model set is non-empty. This might fail in the presence of ever better choices, without ideal ones; the problem is addressed by the limit versions.

(8) (PR) is an infinitary version of one half of the deduction theorem: Let T stand for ϕ, T' for ψ, and $\phi \wedge \psi \mid\!\sim \sigma$, so $\phi \mid\!\sim \psi \rightarrow \sigma$, but $(\psi \rightarrow \sigma) \wedge \psi \vdash \sigma$.

(9) (CUM) (whose more interesting half in our context is (CM)) may best be seen as normal use of lemmas: We have worked hard and found some lemmas. Now we can take a rest, and come back again with our new lemmas. Adding them to the axioms will neither add new theorems, nor prevent old ones to hold. (This is, of course, a meta-level argument concerning an object level rule. But also object level rules should - at least generally - have an intuitive justification, which will then come from a meta-level argument.)

2.2.1 Countably many disjoint sets

This might be the right place to add the following short remark on formula sets.

We show here that - independent of the cardinality of the language - one can define only countably many inconsistent formulas.

The question is due to D.Makinson (personal communication).

Example 2.2.2

There is a countably infinite set of formulas s.t. the defined model sets are pairwise disjoint.

Let $p_i : i \in \omega$ be propositional variables.

Consider $\phi_i := \bigwedge\{\neg p_j : j < i\} \wedge p_i$ for $i \in \omega$.

Obviously, $M(\phi_i) \neq \emptyset$ for all i.

Let $i < i'$, we show $M(\phi_i) \cap M(\phi_{i'}) = \emptyset$. $M(\phi_{i'}) \models \neg p_i$, $M(\phi_i) \models p_i$.

$\square$

Fact 2.2.3

Any set X of consistent formulas with pairwise disjoint model sets is at most countable.

Proof

Let such X be given.

(1) We may assume that X consists of conjunctions of propositional variables or their negations.

Proof: Re-write all $\phi \in X$ as disjunctions of conjunctions ϕ_j. At least one of the conjunctions ϕ_j is consistent. Replace ϕ by one such ϕ_j. Consistency is preserved, as is pairwise disjointness.

(2) Let X be such a set of formulas. Let $X_i \subseteq X$ be the set of formulas in X with length i, i.e. a consistent conjunction of i many propositional variables or their negations, $i > 0$.

As the model sets for X are pairwise disjoint, the model sets for all $\phi \in X_i$ have to be disjoint.

(3) It suffices now to show that each X_i is at most countable, we even show that each X_i is finite.

Proof by induction:

Consider $i = 1$. Let $\phi, \phi' \in X_1$. Let ϕ be p or $\neg p$. If ϕ' is not $\neg\phi$, then ϕ and ϕ' have a common model. So one must be p, the other $\neg p$. But these are all possibilities, so $card(X_1)$ is finite.

Let the result be shown for $k < i$.

Consider now X_i. Take arbitrary $\phi \in X_i$. Wlog, $\phi = p_1 \wedge \ldots \wedge p_i$. Take arbitrary $\phi' \neq \phi$. As $M(\phi) \cap M(\phi') = \emptyset$, ϕ' must be a conjunction containing one of $\neg p_k$, $1 \leq k \leq i$. Consider now $X_{i,k} := \{\phi' \in X_i : \phi' \text{ contains } \neg p_k\}$. Thus $X_i = \{\phi\} \cup \bigcup\{X_{i,k} : 1 \leq k \leq i\}$. Note that all $\psi, \psi' \in X_{i,k}$ agree on $\neg p_k$, so the situation in $X_{i,k}$ is isomorphic to X_{i-1}. So, by induction hypothesis, $card(X_{i,k})$ is finite, as all $\phi' \in X_{i,k}$ have to be mutually inconsistent. Thus, $card(X_i)$ is finite. (Note that we did not use the fact that elements from different $X_{i,k}$, $X_{i,k'}$ also have to be mutually inconsistent, our rough proof suffices.)

□

Note that the proof depends very little on logic. We needed normal forms, and used 2 truth values. Obviously, we can easily generalize to finitely many truth values.

2.2.2 Introduction to many-valued logics

In 2-valued logic, we have a correspondence between sets and logic, e.g., we can speak abut the set of models of a formula. This has now to be replaced by a many-valued function (or, alternatively, but generally not pursued here, by many-valued sets).

Motivation

Preferential logics offer, indirectly, 4 truth values: classically true and false and defeasibly true and false. Inheritance systems offer, through specificity, arbitrarily many truth values, with a partial order. Finite Goedel logics offer arbitrarily many truth values, with a total order. This motivates in our context to consider the following:

Definition 2.2.7

(1) We assume here a finite set of truth values, V, to be given, with a partial order $\leq$. We assume that a minimal and a maximal element exist, which will be denoted 0 and 1, TRUE and FALSE, min and max, depending on context. In many cases we will assume that sup and inf (equivalently, max and min, as V is supposed to be finite) exist for any subset of V. This will not always be necessary, but often it will be convenient.

Inf will correspond to classical $\wedge$, sup to classical $\vee$. Given $x \in V$, $\boldsymbol{C}x$ will be $inf\{y \in V : sup\{x, y\} = 1\}$, it corresponds to classical $\neg$. We will not assume that classical $\neg$ is always part of the language.

(2) A model is a function $m : L \rightarrow V$.

In classical logic, a formula ϕ defines a model set $M(\phi) \subseteq M$, equivalently a function $f_\phi : M \rightarrow \{0, 1\}$ with $f_\phi(m) = 1 :\Leftrightarrow m \models \phi$. A straightforward generalization is to define in the many-valued case $f_\phi : M \rightarrow V$.

This definition should respect the following postulates:

(2.1) $f_p(m) = m(p)$ for $p \in L$.

This postulate is the basis for a seemingly trivial property, which has far-reaching consequences: If ϕ contains only variables in $L' \subseteq L$, then its truth value is the same in $L-$models and $L'-$models, whenever they agree on L'. Of course, validity of the operators has to be truth functional, and again not to depend on other variables.

(2.2) $f_{\phi\wedge\psi}(m) = inf\{f_\phi(m), f_\psi(m)\}$ and $f_{\phi\vee\psi}(m) = sup\{f_\phi(m), f_\psi(m)\}$.

For a set T of formulas, we define $f_T(m) := inf\{f_\phi(m) : \phi \in T\}$.

(3) In general, a model set corresponds now to an arbitrary function $f : M \rightarrow V$. Such f is called (formula) definable iff there is ϕ such that $f = f_\phi$. The definition of theory definable is analogous.

$\mathcal{P}(M)$ is replaced by V^M, the set of all functions from M to V, $\mathcal{D}$ will denote the set of all definable functions from M to V.

(4) Semantical consequence should respect $\leq$:

(4.1) For $f, g : M \rightarrow V$, we write $f \models g$ or $f \leq g$ iff $\forall m \in M.f(m) \leq g(m)$,

(4.2) we write $\phi \models \psi$ iff $f_\phi \models f_\psi$,

(4.3) and we assume that $\rightarrow$ is compatible with $\models$:

$f_{\phi\rightarrow\psi}(m) = TRUE$ iff $f_\phi(m) \leq f_\psi(m)$, so

$f_{\phi\rightarrow\psi} =$ (constant) TRUE iff $\phi \models \psi$.

2.2.2.1 Definable model functions

In classical logic, the set of definable model sets satisfies certain closure conditions. We will examine them now, and generalize them.

(1) We will assume that there are formulas TRUE and FALSE (by abuse of language) such that $f_{TRUE}(m) = TRUE$ and $f_{FALSE}(m) = FALSE$ for all m, thus we have at least the two definable constant functions TRUE and FALSE, again by abuse of language.

Not necessarily all constant functions are definable.

(2) For each $x \in L$ there is f_x defined by $f_x(m) := m(x)$.

(3) $\mathcal{D}$ is closed under finite sup and finite inf.

(4) $\mathcal{D}$ will not necessarily be closed under complementation:

Given $f \in \mathcal{D}$, the complement $\mathcal{C}(f)$ is defined as above by:

$\mathcal{C}(f)(m) := inf\{v \in V : sup\{v, f(m)\} = TRUE\} = \mathcal{C}(f(m))$.

(5) In classical logic, $\mathcal{D}$ is closed under simplification: If $X \subseteq M$ is a definable model set, $L' \subseteq L$, then $X' := \{m \in M : \exists m' \in X.m' \upharpoonright L' = m \upharpoonright L'\}$ is definable. This is a consequence of the existence of the standard normal forms, consider e.g. the formula $p \wedge q$, with $L = \{p, q\}$, $L' = \{p\}$, then $X' = \{m, m'\}$, where $m(p) = m(q) = 1$, $m'(p) = 1$, $m'(q) = 0$, and the new formula is p. We "neglect" or "forget" q, take the projection. It is also a sufficient condition for syntactic interpolation, see Chapter 4 (page 125). (The following Example 2.2.3 (page 50) shows that two different formulas might have the same model function, but should have different projections.)

We have to define the analogon to X' in many-valued logic.

Note that

- if $m \upharpoonright L' = m' \upharpoonright L'$, then $m \in X' \Leftrightarrow m' \in X'$
- $m \in X'$ iff there is $m' \in X.m \upharpoonright L' = m' \upharpoonright L'$, thus $f_{X'}(m) = sup\{f_X(m') : m \upharpoonright L' = m' \upharpoonright L'\}$

So we impose the same conditions: Let f and $L' \subseteq L$ be given, we look for suitable f'.

(5.1) f' has to be indifferent to $L - L'$: if $m \upharpoonright L' = m' \upharpoonright L'$, then we should have $f'(m) = f'(m')$.

(5.2) $f'(m) = sup\{f(m') : m \upharpoonright L' = m' \upharpoonright L'\}$.

(6) When the model set has additional structure, we can ask whether the resulting model choice functions preserve definability:

(6.1) the case of preferential structures is treated below in Definition 2.3.6 (page 67),

(6.2) we can ask the same question e.g. for modal structures: is the set of all models reachable from some model definable, etc.

Example 2.2.3

This example shows that 2 different formulas ϕ and ϕ' may define the same $f_\phi = f_{\phi'}$, but neglecting a certain variable should give different results. We give two variants.

(1) Set $\phi := p \vee (q \vee \neg q)$, $\psi := \neg p \vee (q \vee \neg q)$. So $f_\phi = f_\psi$, but negleting q should result in p in the first case, in $\neg p$ in the second case.

(2) We work with 3 truth values, 0 for FALSE, 2 for TRUE, $\wedge$ is as usual interpreted by inf. Define two new unary operators $K(x) := 1$ (constant), $M(x) := min\{1, x\}$.

a	b	$\phi = K(a) \wedge b$	$\phi' = K(a) \wedge M(b)$
0	0	0	0
0	1	1	1
0	2	1	1
1	0	0	0
1	1	1	1
1	2	1	1
2	0	0	0
2	1	1	1
2	2	1	1

So they define the same model function $f : M \rightarrow V$. But when we forget about a, the first should just be b, but the second should be $M(b)$.

□

We may consider more systematically other operators, under which the definable model sets should be closed:

(1) constants for each truth value, like $FALSE = p \wedge \neg p$, $TRUE = p \vee \neg p$ in classical logic,

(2) complementation, if the complement is defined on the truth value set. (In classical logic, this is, of course, negation.)

This might for instance be interesting for argumentation, where arguments, or their sources, are the truth values.

(3) functions similar to the basic operations SHL and SHR of computer science: suppose a linear order be given on the truth values $0, \ldots, n$, then $SHR(p) := p + 1$ if $p < n$, and e.g. 0 if $p = n$, etc.

The function J of finite Goedel logics, see Definition 4.4.3 (page 149), has some similarity to such shift operations.

2.2.2.2 Generalization of model sets and (in)essential variables, overview

The Table 2.3 (page 54) also contains further material which will become clearer only later.

The Table 2.3 (page 54) summarizes the situation for the 2-valued and the many-valued case.

2.2.2.3 Interpolation of many valued logics

Definition 2.2.8

(3) Given $f, g, h : M \to V$, we say that h is a semantic interpolant for f and g iff

(3.1) $\forall m \in M(f(m) \leq h(m) \leq g(m))$,

(3.2) $I(f) \cup I(g) \subseteq I(h)$

(4) Given ϕ, ψ, we say that α is a syntactic interpolant for ϕ and ψ iff

(4.1) $\forall m \in M(f_\phi(m) \leq f_\alpha(m) \leq f_\psi(m))$,

(4.2) all variables occuring in α occur also in ϕ and ψ.

(5) The following will be central for constructing a semantical interpolant:

Let $L' \subseteq L$, $m \in M$, $m \upharpoonright L' : L' \to V$ be the restriction of a model m to L', $f : M \to V$, then

$f^+(m \upharpoonright L') := max\{f(m') : m \sim_{L'} m'\}$, the maximal value for any m' which agrees with m on L', and

$f^-(m \upharpoonright L') := min\{f(m') : m \sim_{L'} m'\}$, the minimal value for any m' which agrees with m on L'.

Logical rules, definitions and connections Part I					
Logical rule		Corr.	Model set	Corr.	Size Rules
Basics					
(SC) Supraclassicality $\alpha \vdash \beta \Rightarrow \alpha \mid\!\sim \beta$ (REF) Reflexivity $T \cup \{\alpha\} \mid\!\sim \alpha$	(SC) $\overline{T} \subseteq \overline{\overline{T}}$	$\Rightarrow$ $\Leftarrow$	$(\mu \subseteq)$ $f(X) \subseteq X$	trivial	(Opt)
(LLE) Left Logical Equivalence $\vdash \alpha \leftrightarrow \alpha', \alpha \mid\!\sim \beta \Rightarrow$ $\alpha' \mid\!\sim \beta$	(LLE) $\overline{T} = \overline{T'} \Rightarrow \overline{\overline{T}} = \overline{\overline{T'}}$		(trivally true)		
(RW) Right Weakening $\alpha \mid\!\sim \beta, \vdash \beta \rightarrow \beta' \Rightarrow$ $\alpha \mid\!\sim \beta'$	(RW) $T \mid\!\sim \beta, \vdash \beta \rightarrow \beta' \Rightarrow$ $T \mid\!\sim \beta'$		(upward closure)	trivial	(iM)
(wOR) $\alpha \mid\!\sim \beta,\ \alpha' \vdash \beta \Rightarrow$ $\alpha \vee \alpha' \mid\!\sim \beta$	(wOR) $\overline{\overline{T}} \cap \overline{T'} \subseteq \overline{\overline{T \vee T'}}$	$\Rightarrow$ $\Leftarrow$	(μwOR) $f(X \cup Y) \subseteq f(X) \cup Y$	$\Leftrightarrow$	$(eM\mathcal{I})$
$(disjOR)$ $\alpha \vdash \neg\alpha',\ \alpha \mid\!\sim \beta,$ $\alpha' \mid\!\sim \beta \Rightarrow \alpha \vee \alpha' \mid\!\sim \beta$	$(disjOR)$ $\neg Con(T \cup T') \Rightarrow$ $\overline{\overline{T}} \cap \overline{\overline{T'}} \subseteq \overline{\overline{T \vee T'}}$	$\Rightarrow$ $\Leftarrow$	$(\mu disjOR)$ $X \cap Y = \emptyset \Rightarrow$ $f(X \cup Y) \subseteq f(X) \cup f(Y)$	$\Leftrightarrow$	$(I \cup disj)$
(CP) Consistency Preservation $\alpha \mid\!\sim \bot \Rightarrow \alpha \vdash \bot$	(CP) $T \mid\!\sim \bot \Rightarrow T \vdash \bot$	$\Rightarrow$ $\Leftarrow$	$(\mu\emptyset)$ $f(X) = \emptyset \Rightarrow X = \emptyset$	trivial	(I_1)
			$(\mu\emptyset fin)$ $X \neq \emptyset \Rightarrow f(X) \neq \emptyset$ for finite X		(I_1)
	(AND_1) $\alpha \mid\!\sim \beta \Rightarrow \alpha \not\mid\!\sim \neg\beta$				(I_2)
	(AND_n) $\alpha \mid\!\sim \beta_1, \ldots, \alpha \mid\!\sim \beta_{n-1} \Rightarrow$ $\alpha \not\mid\!\sim (\neg\beta_1 \vee \ldots \vee \neg\beta_{n-1})$				(I_n)
(AND) $\alpha \mid\!\sim \beta, \alpha \mid\!\sim \beta' \Rightarrow$ $\alpha \mid\!\sim \beta \wedge \beta'$	(AND) $T \mid\!\sim \beta, T \mid\!\sim \beta' \Rightarrow$ $T \mid\!\sim \beta \wedge \beta'$		(closure under finite intersection)	trivial	(I_ω)
(CCL) Classical Closure	(CCL) $\overline{\overline{T}}$ classically closed		(trivally true)	trivial	$(iM) + (I_\omega)$
(OR) $\alpha \mid\!\sim \beta, \alpha' \mid\!\sim \beta \Rightarrow$ $\alpha \vee \alpha' \mid\!\sim \beta$	(OR) $\overline{\overline{T}} \cap \overline{\overline{T'}} \subseteq \overline{\overline{T \vee T'}}$	$\Rightarrow$ $\Leftarrow$	(μOR) $f(X \cup Y) \subseteq f(X) \cup f(Y)$	$\Leftrightarrow$	$(eM\mathcal{I}) + (I_\omega)$
$\overline{\overline{\alpha \wedge \alpha'}} \subseteq \overline{\overline{\overline{\alpha}} \cup \{\alpha'\}}$	(PR) $\overline{\overline{T \cup T'}} \subseteq \overline{\overline{\overline{T}} \cup T'}$	$\Rightarrow$ $\Leftarrow (\mu dp) + (\mu \subseteq)$ $\not\Leftarrow -(\mu dp)$ $\Leftarrow (\mu \subseteq)$ $T' = \phi$	(μPR) $X \subseteq Y \Rightarrow$ $f(Y) \cap X \subseteq f(X)$	$\Leftrightarrow$	$(eM\mathcal{I}) + (I_\omega)$
		$\Leftarrow$ $T' = \phi$	$(\mu PR')$ $f(X) \cap Y \subseteq f(X \cap Y)$		
(CUT) $T \mid\!\sim \alpha; T \cup \{\alpha\} \mid\!\sim \beta \Rightarrow$ $T \mid\!\sim \beta$	(CUT) $T \subseteq \overline{T'} \subseteq \overline{\overline{T}} \Rightarrow$ $\overline{\overline{T'}} \subseteq \overline{\overline{T}}$	$\Rightarrow$ $\Leftarrow$	(μCUT) $f(X) \subseteq Y \subseteq X \Rightarrow$ $f(X) \subseteq f(Y)$	$\Leftarrow$ $\not\Rightarrow$	$(eM\mathcal{I}) + (I_\omega)$

Table 2.1: Logical rules, definitions and connections Part I

Logical rules, definitions and connections Part II					
Logical rule		Corr.	Model set	Corr.	Size-Rule
Cumulativity					
(wCM) $\alpha \mid\sim \beta, \alpha' \vdash \alpha, \alpha \wedge \beta \vdash \alpha' \Rightarrow$ $\alpha' \mid\sim \beta$				trivial	$(eM\mathcal{F})$
(CM_2) $\alpha \mid\sim \beta, \alpha \mid\sim \beta' \Rightarrow \alpha \wedge \beta \not\vdash \neg\beta'$					(I_2)
(CM_n) $\alpha \mid\sim \beta_1, \ldots, \alpha \mid\sim \beta_n \Rightarrow$ $\alpha \wedge \beta_1 \wedge \ldots \wedge \beta_{n-1} \not\vdash \neg\beta_n$					(I_n)
(CM) Cautious Monotony $\alpha \mid\sim \beta, \alpha \mid\sim \beta' \Rightarrow$ $\alpha \wedge \beta \mid\sim \beta'$	(CM) $T \subseteq \overline{\overline{T'}} \subseteq \overline{\overline{T}} \Rightarrow$ $\overline{\overline{T}} \subseteq \overline{\overline{T'}}$	$\Rightarrow$ $\Leftarrow$	(μCM) $f(X) \subseteq Y \subseteq X \Rightarrow$ $f(Y) \subseteq f(X)$	$\Leftrightarrow$	$(\mathcal{M}^+_\omega)(4)$
or $(ResM)$ Restricted Monotony $T \mid\sim \alpha, \beta \Rightarrow T \cup \{\alpha\} \mid\sim \beta$		$\Rightarrow$ $\Leftarrow$	$(\mu ResM)$ $f(X) \subseteq A \cap B \Rightarrow$ $f(X \cap A) \subseteq B$		
(CUM) Cumulativity $\alpha \mid\sim \beta \Rightarrow$ $(\alpha \mid\sim \beta' \Leftrightarrow \alpha \wedge \beta \mid\sim \beta')$	(CUM) $T \subseteq \overline{\overline{T'}} \subseteq \overline{\overline{T}} \Rightarrow$ $\overline{\overline{T}} = \overline{\overline{T'}}$	$\Rightarrow$ $\Leftarrow$	(μCUM) $f(X) \subseteq Y \subseteq X \Rightarrow$ $f(Y) = f(X)$	$\Leftarrow$ $\not\Rightarrow$	$(eM\mathcal{I}) + (I_\omega) + (\mathcal{M}^+_\omega)(4)$
	$(\subseteq\supseteq)$ $T \subseteq \overline{\overline{T'}}, T' \subseteq \overline{\overline{T}} \Rightarrow$ $\overline{\overline{T'}} = \overline{\overline{T}}$	$\Rightarrow$ $\Leftarrow$	$(\mu \subseteq\supseteq)$ $f(X) \subseteq Y, f(Y) \subseteq X \Rightarrow$ $f(X) = f(Y)$	$\Leftarrow$ $\not\Rightarrow$	$(eM\mathcal{I}) + (I_\omega) + (eM\mathcal{F})$
Rationality					
$(RatM)$ Rational Monotony $\alpha \mid\sim \beta, \alpha \not\mid\sim \neg\beta' \Rightarrow$ $\alpha \wedge \beta' \mid\sim \beta$	$(RatM)$ $Con(T \cup \overline{\overline{T'}}), T \vdash T' \Rightarrow$ $\overline{\overline{T}} \supseteq \overline{\overline{\overline{T'} \cup T}}$	$\Rightarrow$ $\Leftarrow (\mu dp)$ $\not\Leftarrow -(\mu dp)$ $\Leftarrow T = \phi$	$(\mu RatM)$ $X \subseteq Y, X \cap f(Y) \neq \emptyset \Rightarrow$ $f(X) \subseteq f(Y) \cap X$	$\Leftrightarrow$	$(\mathcal{M}^{++})$
	$(RatM =)$ $Con(T \cup \overline{\overline{T'}}), T \vdash T' \Rightarrow$ $\overline{\overline{T}} = \overline{\overline{\overline{T'} \cup T}}$	$\Rightarrow$ $\Leftarrow (\mu dp)$ $\not\Leftarrow -(\mu dp)$ $\Leftarrow T = \phi$	$(\mu =)$ $X \subseteq Y, X \cap f(Y) \neq \emptyset \Rightarrow$ $f(X) = f(Y) \cap X$		
	$(Log =')$ $Con(\overline{\overline{T'}} \cup T) \Rightarrow$ $\overline{\overline{T \cup T'}} = \overline{\overline{\overline{T'} \cup T}}$	$\Rightarrow$ $\Leftarrow (\mu dp)$ $\not\Leftarrow -(\mu dp)$ $\Leftarrow T = \phi$	$(\mu =')$ $f(Y) \cap X \neq \emptyset \Rightarrow$ $f(Y \cap X) = f(Y) \cap X$		
(DR) $\alpha \vee \beta \mid\sim \gamma \Rightarrow$ $\alpha \mid\sim \gamma$ or $\beta \mid\sim \gamma$	$(Log \parallel)$ $\overline{\overline{T \vee T'}}$ is one of $\overline{\overline{T}}$, or $\overline{\overline{T'}}$, or $\overline{\overline{T}} \cap \overline{\overline{T'}}$ (by (CCL))	$\Rightarrow$ $\Leftarrow$	$(\mu \parallel)$ $f(X \cup Y)$ is one of $f(X)$, $f(Y)$ or $f(X) \cup f(Y)$		
	$(Log\cup)$ $Con(\overline{\overline{T'}} \cup T), \neg Con(\overline{\overline{T'}} \cup \overline{\overline{T}}) \Rightarrow$ $\neg Con(\overline{\overline{T \vee T'}} \cup T')$	$\Rightarrow (\mu \subseteq) + (\mu =)$ $\Leftarrow (\mu dp)$ $\not\Leftarrow -(\mu dp)$	$(\mu\cup)$ $f(Y) \cap (X - f(X)) \neq \emptyset \Rightarrow$ $f(X \cup Y) \cap Y = \emptyset$		
	$(Log\cup')$ $Con(\overline{\overline{T'}} \cup T), \neg Con(\overline{\overline{T'}} \cup \overline{\overline{T}}) \Rightarrow$ $\overline{\overline{T \vee T'}} = \overline{\overline{T}}$	$\Rightarrow (\mu \subseteq) + (\mu =)$ $\Leftarrow (\mu dp)$ $\not\Leftarrow -(\mu dp)$	$(\mu\cup')$ $f(Y) \cap (X - f(X)) \neq \emptyset \Rightarrow$ $f(X \cup Y) = f(X)$		
			$(\mu \in)$ $a \in X - f(X) \Rightarrow$ $\exists b \in X. a \notin f(\{a, b\})$		

Table 2.2: Logical rules, definitions and connections Part II

Table 2.3: Notation and Definitions

Notation and definitions		
propositional language L, $L' \subseteq L$, propositional variables $s, \ldots$		
	2-valued $\{0,1\}$	many-valued $(V, \leq)$
definability of f	$\exists \phi : f_\phi = f$	
model m M set of all $L-$models	$m : L \to \{0,1\}$	$m : L \to V$
(for $\Gamma \subseteq M$) $\Gamma \restriction L'$	$\Gamma \restriction L' := \{m \restriction L' : m \in \Gamma\}$	
$m \restriction L'$	like m, but restricted to L'	
$m \sim_{L'} m'$	$m \sim_{L'} m'$ iff $\forall s \in L'.m(s) = m'(s)$	
model "*set*″ of formula ϕ	$M(\phi) \subseteq M$, $f_\phi : M \to \{0,1\}$	$f_\phi : M \to V$
semantic equivalence of ϕ, ψ	$f_\phi = f_\psi$	
general model set	$M' \subseteq M$, $f : M \to \{0,1\}$	$f : M \to V$
f insensitive to L'	$\forall m, m' \in M.(m \sim_{L-L'} m' \Rightarrow f(m) = f(m'))$	
(ir)relevant	$s \in L$ is irrelevant for f iff f is insensitive to s, $I(f) := \{s \in L : s$ is irrelevant for $f\}$, $R(f) := L - I(f)$ $I(\phi) := I(f_\phi)$, $R(\phi) := R(f_\phi)$	
$f^+(m \restriction L')$, $f^-(m \restriction L')$	$f^+(m \restriction L') := max\{f(m') : m' \in M, m \sim_{L'} m'\}$ $f^-(m \restriction L') := min\{f(m') : m' \in M, m \sim_{L'} m'\}$	
$f \leq g$	$\forall m \in M.f(m) \leq g(m)$	

2.3 Preferential structures

An important part of these notes is motivated or concerns directly preferential structures, which we now define.

2.3.1 The minimal variant

Definition 2.3.1

Fix $U \neq \emptyset$, and consider arbitrary X. Note that this X has not necessarily anything to do with U, or $\mathcal{U}$ below. Thus, the functions $\mu_{\mathcal{M}}$ below are in principle functions from V to V - where V is the set theoretical universe we work in.

Note that we work here often with copies of elements (or models). In other areas of logic, most authors work with valuation functions. Both definitions - copies or valuation functions - are equivalent, a copy $\langle x, i\rangle$ can be seen as a state $\langle x, i\rangle$ with valuation x. In the beginning of research on preferential structures, the notion of copies was widely used, whereas e.g., [KLM90] used that of valuation functions. There is perhaps a weak justification of the former terminology. In modal logic, even if two states have the same valid classical formulas, they might still be distinguishable by their valid modal formulas. But this depends on the fact that modality is in the object language. In most work on preferential stuctures, the consequence relation is outside the object language, so different states with same valuation are in a stronger sense copies of each other.

(1) *Preferential models* or *structures*.

(1.1) The version without copies:

A pair $\mathcal{M} := \langle U, \prec\rangle$ with U an arbitrary set, and $\prec$ an arbitrary binary relation on U is called a *preferential model* or *structure*.

(1.2) The version with copies :

A pair $\mathcal{M} := \langle \mathcal{U}, \prec\rangle$ with $\mathcal{U}$ an arbitrary set of pairs, and $\prec$ an arbitrary binary relation on $\mathcal{U}$ is called a *preferential model* or *structure*.

If $\langle x, i\rangle \in \mathcal{U}$, then x is intended to be an element of U, and i the index of the copy.

We sometimes also need copies of the relation $\prec$. We will then replace $\prec$ by one or several arrows α attacking non-minimal elements, e.g., $x \prec y$ will be written $\alpha : x \to y$, $\langle x, i\rangle \prec \langle y, i\rangle$ will be written $\alpha : \langle x, i\rangle \to \langle y, i\rangle$, and finally we might have $\langle \alpha, k\rangle : x \to y$ and $\langle \alpha, k\rangle : \langle x, i\rangle \to \langle y, i\rangle$, etc.

(2) *Minimal elements*, the functions $\mu_{\mathcal{M}}$

(2.1) The version without copies:

Let $\mathcal{M} := \langle U, \prec\rangle$, and define

$\mu_{\mathcal{M}}(X) := \{x \in X : x \in U \wedge \neg\exists x' \in X \cap U.x' \prec x\}.$

$\mu_{\mathcal{M}}(X)$ is called the set of *minimal elements* of X (in $\mathcal{M}$).

Thus, $\mu_{\mathcal{M}}(X)$ is the set of elements such that there is no smaller one in X.

(2.2) The version with copies:
Let $\mathcal{M} := \langle \mathcal{U}, \prec \rangle$ be as above. Define
$\mu_{\mathcal{M}}(X) := \{x \in X : \exists \langle x, i \rangle \in \mathcal{U}.\neg\exists \langle x', i' \rangle \in \mathcal{U}(x' \in X \wedge \langle x', i' \rangle' \prec \langle x, i \rangle)\}.$
Thus, $\mu_{\mathcal{M}}(X)$ is the projection on the first coordinate of the set of elements such that there is no smaller one in X.
Again, by abuse of language, we say that $\mu_{\mathcal{M}}(X)$ is the set of *minimal elements* of X in the structure. If the context is clear, we will also write just μ.
We sometimes say that $\langle x, i \rangle$ "*kills*" or "*minimizes*" $\langle y, j \rangle$ if $\langle x, i \rangle \prec \langle y, j \rangle$. By abuse of language we also say a set X *kills* or *minimizes* a set Y if for all $\langle y, j \rangle \in \mathcal{U}$, $y \in Y$ there is $\langle x, i \rangle \in \mathcal{U}$, $x \in X$ s.t. $\langle x, i \rangle \prec \langle y, j \rangle$.
$\mathcal{M}$ is also called *injective* or 1-copy , iff there is always at most one copy $\langle x, i \rangle$ for each x. Note that the existence of copies corresponds to a non-injective labelling function - as is often used in nonclassical logic, e.g., modal logic.

We say that $\mathcal{M}$ is *transitive*, *irreflexive*, etc., iff $\prec$ is.

Note that $\mu(X)$ might well be empty, even if X is not.

Definition 2.3.2

We define the consequence relation of a preferential structure for a given propositional language $\mathcal{L}$.

(1) (1.1) If m is a classical model of a language $\mathcal{L}$, we say by abuse of language
$\langle m, i \rangle \models \phi$ iff $m \models \phi$,
and if X is any set of such pairs, that
$X \models \phi$ iff for all $\langle m, i \rangle \in X$ $m \models \phi$.

(1.2) If $\mathcal{M}$ is a preferential structure, and X is a set of $\mathcal{L}-$models for a classical propositional language $\mathcal{L}$, or a set of pairs $\langle m, i \rangle$, where the m are such models, we call $\mathcal{M}$ a *classical preferential structure* or *model*.

(2) *Validity* in a preferential structure, or the *semantical consequence relation* defined by such a structure:

Let $\mathcal{M}$ be as above.

We define:

$T \models_{\mathcal{M}} \phi$ iff $\mu_{\mathcal{M}}(M(T)) \models \phi$, i.e., $\mu_{\mathcal{M}}(M(T)) \subseteq M(\phi)$.

(3) $\mathcal{M}$ will be called *definability preserving* iff for all $X \in \boldsymbol{D}_{\mathcal{L}}$ $\mu_{\mathcal{M}}(X) \in \boldsymbol{D}_{\mathcal{L}}$.

As $\mu_{\mathcal{M}}$ is defined on $\boldsymbol{D}_{\mathcal{L}}$, but need by no means always result in some new definable set, this is (and reveals itself as a quite strong) additional property.

Definition 2.3.3

Let $\mathcal{Y} \subseteq \mathcal{P}(U)$. (In applications to logic, $\mathcal{Y}$ will be $\boldsymbol{D}_{\mathcal{L}}$.)

A preferential structure $\mathcal{M}$ is called $\mathcal{Y}-$smooth iff for every $X \in \mathcal{Y}$ every element $x \in X$ is either minimal in X or above an element, which is minimal in X. More precisely:

(1) The version without copies:

If $x \in X \in \mathcal{Y}$, then either $x \in \mu(X)$ or there is $x' \in \mu(X).x' \prec x$.

(2) The version with copies:

If $x \in X \in \mathcal{Y}$, and $\langle x,i \rangle \in \mathcal{U}$, then either there is no $\langle x',i' \rangle \in \mathcal{U}$, $x' \in X$, $\langle x',i' \rangle \prec \langle x,i \rangle$ or there is $\langle x',i' \rangle \in \mathcal{U}$, $\langle x',i' \rangle \prec \langle x,i \rangle$, $x' \in X$, s.t. there is no $\langle x'',i'' \rangle \in \mathcal{U}$, $x'' \in X$, with $\langle x'',i'' \rangle \prec \langle x',i' \rangle$.

(Writing down all details here again might make it easier to read applications of the definition later on.)

When considering the models of a language $\mathcal{L}$, $\mathcal{M}$ will be called *smooth* iff it is $\boldsymbol{D}_{\mathcal{L}}-$smooth ; $\boldsymbol{D}_{\mathcal{L}}$ is the default.

Obviously, the richer the set $\mathcal{Y}$ is, the stronger the condition $\mathcal{Y}-$smoothness will be.

A remark for the intuition: Smoothness is perhaps best motivated through Gabbay's concept of reactive diagrams, see, e.g., [Gab04] and [Gab08], and also [GS08c], [GS08f]. In this concept, smaller, or "better" elements attack bigger, or "less good" elements. But when a attacks b, and b attacks c, then one might consider the attack of b against c weakened by the attack of a against b. In a smooth structure, for every attack against some element x, there is also an uncontested attack against x, as it originates in an element y, which is not attacked itself.

Fact 2.3.1

Let $\prec$ be an irreflexive, binary relation on X, then the following two conditions are equivalent:

(1) There is Ω and an irreflexive, total, binary relation $\prec'$ on Ω and a function $f : X \to \Omega$ s.t. $x \prec y \Leftrightarrow f(x) \prec' f(y)$ for all $x, y \in X$.

(2) Let $x, y, z \in X$ and $x \perp y$ wrt. $\prec$ (i.e., neither $x \prec y$ nor $y \prec x$), then $z \prec x \Rightarrow z \prec y$ and $x \prec z \Rightarrow y \prec z$.

Proof

(1) $\Rightarrow$ (2): Let $x \perp y$, thus neither $f(x) \prec' f(y)$ nor $f(y) \prec' f(x)$, but then $f(x) = f(y)$. Let now $z \prec x$, so $f(z) \prec' f(x) = f(y)$, so $z \prec y$. $x \prec z \Rightarrow y \prec z$ is similar.

(2) $\Rightarrow$ (1): For $x \in X$ let $[x] := \{x' \in X : x \perp x'\}$, and $\Omega := \{[x] : x \in X\}$. For $[x], [y] \in \Omega$ let $[x] \prec' [y] :\Leftrightarrow x \prec y$. This is well-defined: Let $x \perp x'$, $y \perp y'$ and $x \prec y$, then $x \prec y'$ and $x' \prec y'$. Obviously, $\prec'$ is an irreflexive, total binary relation. Define $f : X \to \Omega$ by $f(x) := [x]$, then $x \prec y \Leftrightarrow [x] \prec' [y] \Leftrightarrow f(x) \prec' f(y)$. $\square$

Definition 2.3.4

We call an irreflexive, binary relation $\prec$ on X, which satisfies (1) (equivalently (2)) of Fact 2.3.1 (page 57) , ranked . By abuse of language, we also call a preferential structure $\langle X, \prec \rangle$ ranked, iff $\prec$ is.

We quote from [Sch04] the following summary for preferential structures:

Table 2.4 (page 59), "Preferential representation", summarizes the more difficult half of a full representation result for preferential structures. It shows equivalence between certain abstract conditions for model choice functions and certain preferential structures. They are shown in the respective representation theorems.

"singletons" means that the domain must contain all singletons, "1 copy" or "$\geq$ 1 copy" means that the structure may contain only 1 copy for each point, or several, "$(\mu\emptyset)$" etc. for the preferential structure mean that the $\mu-$function of the structure has to satisfy this property.

We call a characterization "normal" iff it is a universally quantified boolean combination (of any fixed, but perhaps infinite, length) of rules of the usual form. We do not go into details here.

In the second column from the left "$\Rightarrow$" means, for instance for the smooth case, that for any $\mathcal{Y}$ closed under finite unions, and any choice function f which satisfies the conditions in the left hand column, there is a (here $\mathcal{Y}-$smooth) preferential structure $\mathcal{X}$ which represents it, i.e., for all $Y \in \mathcal{Y}$ $f(Y) = \mu_{\mathcal{X}}(Y)$, where $\mu_{\mathcal{X}}$ is the model choice function of the structure $\mathcal{X}$. The inverse arrow $\Leftarrow$ means that the model choice function for any smooth $\mathcal{X}$ defined on such $\mathcal{Y}$ will satisfy the conditions on the left.

For more detail on preferential logics and size, the reader is referred to, e.g., [GS08c] and [GS09a].

In Section 3.3 (page 104), we will generalize the concept of a preferential structure to logics with more than two truth values, see Definition 2.3.6 (page 67) there.

2.3.1.1 New material on the minimal variant of preferential structures (Bubble structures)

We discuss now a variant of preferential structures which will prove to be useful. We call them bubble structures, see Diagram 2.3.1 (page 60) for illustration.

Table 2.4: Preferential representation

Preferential representation				
$\mu-$ function		Pref.Structure		Logic
$(\mu \subseteq)$	$\Leftrightarrow$	reactive	$\Leftrightarrow$	$(LLE)+(CCL)+$ (SC)
$(\mu \subseteq)+(\mu CUM)$	$\Rightarrow$ $(\cap)$	reactive + essentially smooth		
$(\mu \subseteq)+(\mu \subseteq\supseteq)$	$\Rightarrow$	reactive + essentially smooth	$\Leftrightarrow$	$(LLE)+(CCL)+$ $(SC)+(\subseteq\supseteq)$
$(\mu \subseteq)+(\mu CUM)+(\mu \subseteq\supseteq)$	$\Leftarrow$	reactive + essentially smooth		
$(\mu \subseteq)+(\mu PR)$	$\Leftarrow$	general	$\Rightarrow$ (μdp)	$(LLE)+(RW)+$ $(SC)+(PR)$
	$\Rightarrow$		$\Leftarrow$	
			$\not\Rightarrow$ without (μdp)	
			$\not\Leftrightarrow$ without (μdp)	any "normal" characterization of any size
$(\mu \subseteq)+(\mu PR)$	$\Leftarrow$	transitive	$\Rightarrow$ (μdp)	$(LLE)+(RW)+$ $(SC)+(PR)$
	$\Rightarrow$		$\Leftarrow$	
			$\not\Rightarrow$ without (μdp)	
			$\Leftrightarrow$ without (μdp)	using "small" exception sets
$(\mu \subseteq)+(\mu PR)+(\mu CUM)$	$\Leftarrow$	smooth	$\Rightarrow$ (μdp)	$(LLE)+(RW)+$ $(SC)+(PR)+$ (CUM)
	$\Rightarrow$ $(\cup)$		$\Leftarrow$ $(\cup)$	
	$\not\Rightarrow$ without $(\cup)$		$\not\Rightarrow$ without (μdp)	
$(\mu \subseteq)+(\mu PR)+(\mu CUM)$	$\Leftarrow$	smooth+transitive	$\Rightarrow$ (μdp)	$(LLE)+(RW)+$ $(SC)+(PR)+$ (CUM)
	$\Rightarrow$ $(\cup)$		$\Leftarrow$ $(\cup)$	
			$\not\Rightarrow$ without (μdp)	
			$\Leftrightarrow$ without (μdp)	using "small" exception sets
$(\mu \subseteq)+(\mu =)+(\mu PR)+$ $(\mu =')+(\mu \parallel)+(\mu\cup)+$ $(\mu\cup')+(\mu \in)+(\mu RatM)$	$\Leftarrow$	ranked, ≥ 1 copy		
$(\mu \subseteq)+(\mu =)+(\mu PR)+$ $(\mu\cup)+(\mu \in)$	$\not\Rightarrow$	ranked		
$(\mu \subseteq)+(\mu =)+(\mu\emptyset)$	$\Leftrightarrow$, $(\cup)$	ranked, 1 copy + $(\mu\emptyset)$		
$(\mu \subseteq)+(\mu =)+(\mu\emptyset)$	$\Leftrightarrow$, $(\cup)$	ranked, smooth, 1 copy + $(\mu\emptyset)$		
$(\mu \subseteq)+(\mu =)+(\mu\emptyset fin)+$ $(\mu \in)$	$\Leftrightarrow$, $(\cup)$, singletons	ranked, smooth, ≥ 1 copy + $(\mu\emptyset fin)$		
$(\mu \subseteq)+(\mu PR)+(\mu \parallel)+$ $(\mu\cup)+(\mu \in)$	$\Leftrightarrow$, $(\cup)$, singletons	ranked ≥ 1 copy	$\not\Rightarrow$ without (μdp)	$(RatM)$, $(RatM =)$, $(Log\cup)$, $(Log\cup')$
			$\not\Leftrightarrow$ without (μdp)	any "normal" characterization of any size

The basic idea is as follows: We have one global structure, and parts of it, "bubbles", behave in a uniform way. Thus, with respect to the "outside world", what is inside an individual bubble is indistinguishable.

Formally, if x is outside a given bubble, and b, b' are inside the bubble, then $x \prec b$ iff $x \prec b'$, and $b \prec x$ iff $b' \prec x$. Thus, seen from the outside, a bubble behaves like a layer in a ranked structure. But we do not require the inside of the bubble to consist of only incomparable elements, like a layer in a ranked structure. Thus, ranked structures are special cases of bubble structures, with the layers being the bubbles.

We may identify the bubbles with single points, and, as long as we do not look into the bubbles, we have just a usual preferential structure.

We can see the whole structure as an abstraction, where details are encapsulated in the bubbles. We can also work with different languages, a "global" language for the big structure, and a (or several) sublanguages which are used only inside the bubbles. When all bubbles have the same internal structure, we can also push this structure into the truth values, see Section 3.3 (page 104) . The basic idea is, of course, another expression of modularity, where we try to isolate different aspects of reasoning as much as possible, to simplify the task - divide et impera!

We will not give a full formal representation result here, but point out the important ingredients. The cases to consider (different sublanguages, cumulativity or not inside/outside the bubbles, etc.) may be quite varied, and we leave it to future research to elaborate the details. We turn now to a list of things to consider.

(1) Copies of the same element should probably be in the same bubble.

(2) We will probably work with one copy of each bubble, so we add a 1-copy condition to the global structure, see [Sch04] for a discussion.

(3) Singletons without copies can always be considered to be bubbles.

(4) Given a global structure, the decomposition into bubbles is usually not unique: Take a linear order, then all elements of any interval $[a, b]$ behave to the outside world in the same way, so any such interval can be considered a bubble. But, of course, we want bubbles to be disjoint, or, at least, forming systems of bubbles, superbubbles, etc.

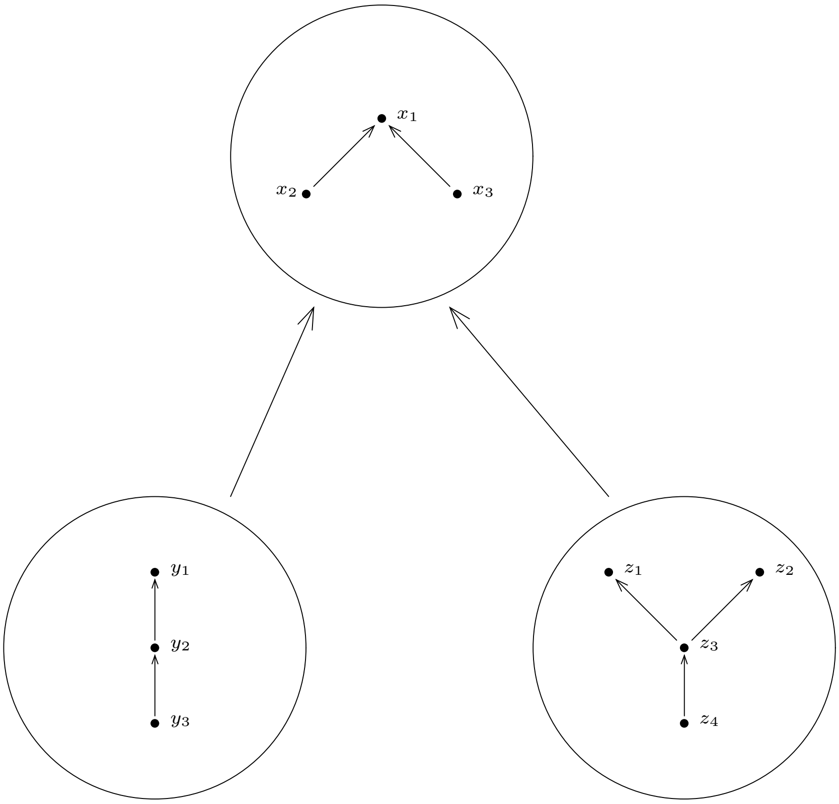
(5) Similar to ranked structures, any element of a bubble replaces any other for the order relation:

Suppose $A \cap B = \emptyset$, B a bubble, then $\mu(A \cup B) = \mu(A) \cup \mu(B)$, or $= \mu(A)$, or $= \mu(B)$, provided the relation is transitive.

For a counterexample to the non-transitive case, consider $b \prec a \prec b'$, but not $b \prec b'$, with $B = \{b, b'\}$, $A = \{a\}$, then $\mu(B) = \{b, b'\}$, $\mu(A \cup B) = \{b\}$.

(6) The full rankedness conditions like $A \subseteq B$, $\mu(B) \cap A \neq \emptyset \Rightarrow \mu(A) = \mu(B) \cap A$ are generally too strong, as the inside of the bubbles need not consist of incomparable elements.

Diagram 2.3.1



2.3.2 The limit variant

Motivation for the limit variant and for our approach

Distance based semantics give perhaps the clearest motivation for the limit variant. For instance, the Stalnaker/Lewis semantics for counterfactual conditionals defines $\phi > \psi$ to hold in a (classical) model m iff in those models of ϕ, which are closest to m, ψ holds. For this to make sense, we need, of course, a distance d on the model set. We call this approach the minimal variant. Usually, one makes a limit assumption: The set of $\phi-$models closest to m is not empty if ϕ is consistent - i.e., the $\phi-$models are not arranged around m in a way that they come closer and closer, without a minimal distance. This is, of course, a very strong assumption, and which is probably difficult to justify philosophically. It seems to have its only justification in the fact that it avoids degenerate cases, where, in above example, for consistent ϕ $m \models \phi > FALSE$ holds. As such, this assumption is unsatisfactory.

The limit version avoids such assumptions. It will still work in above situation, i.e., when there are not always optimal (closest) elements, it defines what happens when we get “better and better”, i.e. approach the limit (the “best” case).

We will have to define what a suitable “neighbourhood” of the best cases is, in our context, this will roughly be a set of elements which minimizes all other elements, and is downward closed, i.e., contains all elements better than some x already in the set. We call such sets MISE, for minimizing initial segment. We will see (Example 2.3.1 (page 63)) that this definition will not always do what we want it to do, and we will have to impose additional properties.

Essentially, we want MISE sets to reflect the properties of the sets of minimal elements, if they exist. Thus, the set of minimal elements should be a special case of a MISE. But we also want MISE sets to be closed under finite intersection, to have the logical (AND) property, see again Example 2.3.1 (page 63). If our definition is such that its properties are sufficiently close to those of the ideal (the minimal elements), then we will also have the desired algebraic and logical properties, but avoid pathologies originating from the empty set (when there are no best elements) - and this is what we wanted. Of course, our definition still has to correspond to the intuition what an approximation to the ideal case should be.

We give now the basic definitions for the limit version of preferential and ranked preferential structures.

Definition 2.3.5

(1) General preferential structures

(1.1) The version without copies:

Let $\mathcal{M} := \langle U, \prec \rangle$. Define

$Y \subseteq X \subseteq U$ is a minimizing initial segment , or MISE , of X iff:

(a) $\forall x \in X \exists x \in Y . y \preceq x$ - where $y \preceq x$ stands for $x \prec y$ or $x = y$ (i.e., Y is minimizing) and

(b) $\forall y \in Y, \forall x \in X (x \prec y \Rightarrow x \in Y)$ (i.e., Y is downward closed or an initial part).

(1.2) The version with copies:

Let $\mathcal{M} := \langle \mathcal{U}, \prec \rangle$ be as above. Define for $Y \subseteq X \subseteq \mathcal{U}$

Y is a minimizing initial segment, or MISE of X iff:

(a) $\forall \langle x, i \rangle \in X \exists \langle y, j \rangle \in Y . \langle y, j \rangle \preceq \langle x, i \rangle$

and

(b) $\forall\langle y,j\rangle \in Y, \forall\langle x,i\rangle \in X$ $(\langle x,i\rangle \prec \langle y,j\rangle \Rightarrow \langle x,i\rangle \in Y)$.

(1.3) For $X \subseteq \mathcal{U}$, let $\Lambda(X)$ be the set of MISE of X.

(1.4) We say that a set $\mathcal{X}$ of MISE is cofinal in another set of MISE $\mathcal{X}'$ (for the same base set X) iff for all $Y' \in \mathcal{X}'$, there is $Y \in \mathcal{X}$, $Y \subseteq Y'$.

(1.5) A MISE X is called definable iff $\{x : \exists i.\langle x,i\rangle \in X\} \in \boldsymbol{D}_{\mathcal{L}}$.

(1.6) $T \models_{\mathcal{M}} \phi$ iff there is $Y \in \Lambda(\mathcal{U} \upharpoonright M(T))$ such that $Y \models \phi$.

($\mathcal{U} \upharpoonright M(T) := \{\langle x,i\rangle \in \mathcal{U} : x \in M(T)\}$ - if there are no copies, we simplify in the obvious way.)

(2) Ranked preferential structures

In the case of ranked structures, we may assume without loss of generality that the MISE sets have a particularly simple form:

For $X \subseteq U$ $A \subseteq X$ is MISE iff $X \neq \emptyset$ and $\forall a \in A \forall x \in X (x \prec a \vee x \perp a \Rightarrow x \in A)$. (A is downward and horizontally closed.)

(3) Theory Revision

Recall that we have a distance d on the model set, and are interested in $y \in Y$ which are close to X.

Thus, given X, Y, we define analogously:

$B \subseteq Y$ is MISE iff

(1) $B \neq \emptyset$

(2) there is d' such that $B := \{y \in Y : \exists x \in X.d(x,y) \leq d'\}$ (we could also have chosen $d(x,y) < d'$, this is not important).

And we define $\phi \in T * T'$ iff there is $B \in \Lambda(M(T), M(T'))$ $B \models \phi$.

There are basic problems with the limit in general preferential structures, as we shall see now:

Example 2.3.1

Let $a \prec b$, $a \prec c$, $b \prec d$, $c \prec d$ (but $\prec$ not transitive!), then $\{a,b\}$ and $\{a,c\}$ are such S and S', but there is no $S'' \subseteq S \cap S'$ which is an initial segment. If, for instance, in a and b ψ holds, in a and c ψ', then "in the limit" ψ and ψ' will hold, but not $\psi \wedge \psi'$. This does not seem right. We should not be obliged to give up ψ to obtain ψ'. $\square$

We will therefore require it to be closed under finite intersections, or at least, that if S, S' are such segments, then there must be $S'' \subseteq S \cap S'$ which is also such a segment.

We make this official. Let $\Lambda(X)$ be the set of initial segments of X, then we require:

($\Lambda\cap$) If $A, B \in \Lambda(X)$ then there is $C \subseteq A \cap B$, $C \in \Lambda(X)$.

To familiarize the reader with the limit version, we show two easy but important results.

Fact 2.3.2

(Taken from [Sch04], Fact 3.4.3, Proposition 3.10.16 there, (2a) is new, but only a summary of other properties.)

Let the relation $\prec$ be transitive. The following hold in the limit variant of general preferential structures:

(1) If $A \in \Lambda(Y)$, and $A \subseteq X \subseteq Y$, then $A \in \Lambda(X)$.

(2) If $A \in \Lambda(Y)$, and $A \subseteq X \subseteq Y$, and $B \in \Lambda(X)$, then $A \cap B \in \Lambda(Y)$.

(2a) We summarize to make finitary semantic cumulativity evident: Let $A \in \Lambda(Y)$, $A \subseteq X \subseteq Y$. Then, if $B \in \Lambda(Y)$, $A \cap B \in \Lambda(X)$. Conversely, if $B \in \Lambda(X)$, then $A \cap B \in \Lambda(Y)$.

(3) If $A \in \Lambda(Y)$, $B \in \Lambda(X)$, then there is $Z \subseteq A \cup B$ $Z \in \Lambda(Y \cup X)$.

The following hold in the limit variant of ranked structures without copies, where the domain is closed under finite unions and contains all finite sets.

(4) $A, B \in \Lambda(X) \Rightarrow A \subseteq B$ or $B \subseteq A$,

(5) $A \in \Lambda(X)$, $Y \subseteq X$, $Y \cap A \neq \emptyset \Rightarrow Y \cap A \in \Lambda(Y)$,

(6) $\Lambda' \subseteq \Lambda(X)$, $\bigcap \Lambda' \neq \emptyset \Rightarrow \bigcap \Lambda' \in \Lambda(X)$.

(7) $X \subseteq Y$, $A \in \Lambda(X) \Rightarrow \exists B \in \Lambda(Y).B \cap X = A$

Proof

(1) trivial.

(2)

(2.1) $A \cap B$ is closed in Y : Let $\langle x, i\rangle \in A \cap B$, $\langle y, j\rangle \prec \langle x, i\rangle$, then $\langle y, j\rangle \in A$. If $\langle y, j\rangle \notin X$, then $\langle y, j\rangle \notin A$, *contradiction*. So $\langle y, j\rangle \in X$, but then $\langle y, j\rangle \in B$.

(2.2) $A \cap B$ minimizes Y : Let $\langle a, i\rangle \in Y$.

(a) If $\langle a, i\rangle \in A - B \subseteq X$, then there is $\langle y, j\rangle \prec \langle a, i\rangle$, $\langle y, j\rangle \in B$. Xy closure of A, $\langle y, j\rangle \in A$.

(b) If $\langle a, i\rangle \notin A$, then there is $\langle a', i'\rangle \in A \subseteq X$, $\langle a', i'\rangle \prec \langle a, i\rangle$, continue by (a).

(2a) For the first part, by (2), $A \cap B \in \Lambda(Y)$, so by (1), $A \cap B \in \Lambda(X)$. The second part is just (2).

(3)

Let $Z := \{\langle x, i\rangle \in A$: $\neg\exists\langle b, j\rangle \preceq \langle x, i\rangle.\langle b, j\rangle \in X - B\} \cup \{\langle y, j\rangle \in B$: $\neg\exists\langle a, i\rangle \preceq \langle y, j\rangle.\langle a, i\rangle \in Y - A\}$, where $\preceq$ stands for $\prec$ or $=$.

(3.1) Z minimizes $Y \cup X$: We consider Y, X is symmetrical.

(a) We first show: If $\langle a, k\rangle \in A - Z$, then there is $\langle y, i\rangle \in Z.\langle a, k\rangle \succ \langle y, i\rangle$. Proof: If $\langle a, k\rangle \in A$−Z, then there is $\langle b, j\rangle \preceq \langle a, k\rangle$, $\langle b, j\rangle \in X$−B. Then there is $\langle y, i\rangle \prec \langle b, j\rangle$, $\langle y, i\rangle \in B$. Xut $\langle y, i\rangle \in Z$, too: If not, there would be $\langle a', k'\rangle \preceq \langle y, i\rangle$, $\langle a', k'\rangle \in Y - A$, but $\langle a', k'\rangle \prec \langle a, k\rangle$, contradicting closure of A.

(b) If $\langle a'', k''\rangle \in Y - A$, there is $\langle a, k\rangle \in A$, $\langle a, k\rangle \prec \langle a'', k''\rangle$. If $\langle a, k\rangle \notin Z$, continue with (a).

(3.2) Z is closed in $Y \cup X$: Let then $\langle z, i\rangle \in Z$, $\langle u, k\rangle \prec \langle z, i\rangle$, $\langle u, k\rangle \in Y \cup X$. Suppose $\langle z, i\rangle \in A$ - the case $\langle z, i\rangle \in B$ is symmetrical.

(a) $\langle u, k\rangle \in Y - A$ cannot be, by closure of A.

(b) $\langle u, k\rangle \in X - B$ cannot be, as $\langle z, i\rangle \in Z$, and by definition of Z.

(c) If $\langle u, k\rangle \in A - Z$, then there is $\langle v, l\rangle \preceq \langle u, k\rangle$, $\langle v, l\rangle \in X$−B, so $\langle v, l\rangle \prec \langle z, i\rangle$, contradicting (b).

(d) If $\langle u,k\rangle \in B-Z$, then there is $\langle v,l\rangle \preceq \langle u,k\rangle$, $\langle v,l\rangle \in Y-A$, contradicting (a).

(4) Suppose not, so there are $a \in A-B$, $b \in B-A$. But if $a \perp b$, $a \in B$ and $b \in A$, similarly if $a \prec b$ or $b \prec a$.

(5) As $A \in \Lambda(X)$ and $Y \subseteq X$, $Y \cap A$ is downward and horizontally closed. As $Y \cap A \neq \emptyset$, $Y \cap A$ minimizes Y.

(6) $\bigcap \Lambda'$ is downward and horizontally closed, as all $A \in \Lambda'$ are. As $\bigcap \Lambda' \neq \emptyset$, $\bigcap \Lambda'$ minimizes X.

(7) Set $B := \{b \in Y : \exists a \in A. a \perp b \text{ or } b \leq a\}$

□

We have as immediate logical consequence:

Fact 2.3.3

(Fact 3.4.4 of [Sch04].)

If $\prec$ is transitive, then in the limit variant hold:

(1) (AND) ,

(2) (OR) .

Proof

Let $\mathcal{Z}$ be the structure.

(1) Immediate by Fact 2.3.2 (page 63), (2) - set $A = B$.

(2) Immediate by Fact 2.3.2 (page 63), (3). □

We also have

Fact 2.3.4

(Fact 3.4.5 in [Sch04])

Finite cumulativity holds in transitive limit structures:

If $\phi \mid\sim \psi$, then $\overline{\overline{\phi}} = \overline{\overline{\phi \wedge \psi}}$.

See [Sch04] for a direct proof, or above Fact 2.3.2 (page 63), (2a). □

We repeat now (without proof) our main logical trivialization results on the limit variant of general preferential structures, Proposition 3.4.7 and Proposition 3.10.19 from [Sch04]:

Proposition 2.3.5

(1) Let the relation be transitive. If we consider only formulas on the left of $\mid\sim$, the resulting logic of the limit version can also be generated by the minimal version of a (perhaps different) preferential structure. Moreover, this structure can be chosen smooth.

(2) Let a logic $\phi \mid\sim \psi$ be given by the limit variant of a ranked structure without copies. Then there is a ranked structure, which gives exactly the same logic, but interpreted in the minimal variant.

□

(The negative results for the general not definability preserving minimal case apply also to the general limit case - see Section 5.2.3 in [Sch04] for details.)

2.3.2.1 New material on the limit variant of preferential structures

This short section contains new material on the limit variant of preferential structures - a discussion of the limit variant of higher preferential structures will be presented below, see Section 2.4.2.6 (page 85).

Consider the following analogon to (μPR) $(A \subseteq B \Rightarrow \mu(B) \cap A \subseteq \mu(A))$:

Fact 2.3.6

Let $\prec$ be transitive, $\Lambda(X)$ the MISE systems over X.

Let $A \subseteq B$, $A' \in \Lambda(A) \Rightarrow \exists B' \in \Lambda(B).B' \cap A \subseteq A'$.

Proof

Consider $B' := \{b \in B : b \notin A - A'$ and $\neg \exists b' \in A - A'.b' \prec b\}$. Thus, $B - B' = \{b \in B : b \in A - A'$ or $\exists b' \in A - A'.b' \prec b\}$.

(1) $B' \cap A \subseteq A'$: Trivial.

(2) B' is closed in B : Let $b \in B'$, suppose there is $b' \in B - B'$, $b' \prec b$. $b' \in A - A'$ is excluded by definition, b' such that $\exists b'' \in A - A'.b'' \prec b'$ by transitivity.

(3) B' is minimizing: Let $b \in B - B'$. If $b \in A - A'$, then there is $a \in A'.a \prec b$ by minimization of A by A'. We have to show that $a \in B'$. If not, there must be $b' \in A - A'.b' \prec a$, contradicting closure of A' in A. If b is such that there is $b' \in A - A'.b' \prec b$. Then there has to be $a \in A'$ such that $a \prec b' \prec b$, so $a \prec b$ by transitivity, and we continue as above.

□

We have immediately:

Corollary 2.3.7

$\overline{\overline{\phi \wedge \phi'}} \subseteq \overline{\overline{\overline{\phi}} \cup \{\phi'\}}$.

Proof

Let $\psi \in \overline{\overline{\phi \wedge \phi'}}$, $A := M(\phi \wedge \phi')$, $B := M(\phi)$. So there is $A' \in \Lambda(A).A' \models \psi$, so there is $B' \in \Lambda(B).B' \cap M(\phi') \models \psi$ by Fact 2.3.6 (page 66), so $B' \models \phi' \rightarrow \psi$, so $\phi' \rightarrow \psi \in \overline{\overline{\phi}}$, so $\overline{\overline{\phi}} \cup \phi' \vdash \psi$. □

2.3.3 Preferential structures for many-valued logics

We can, of course, consider for given ϕ the set of models where ϕ has maximal truth value TRUE, and then take the minimal ones as usual. The resulting logic $\mid\!\sim$ then makes $\phi \mid\!\sim \psi$ true, iff the minimal models with value TRUE assign TRUE also to ψ. See Section 5.3.6 (page 203).

But this does not seem to be the adequate way. So we adapt the definition of preferential structures to the many-valued situation.

Definition 2.3.6

Let $\mathcal{L}$ be given with model set M.

Let a binary relation $\prec$ be given on $\mathcal{X}$, where $\mathcal{X}$ is a set of pairs $\langle m, i\rangle$, $m \in M$, i some index as usual. (We use here the assumption that the truth value is independent of indices.)

Let $f : M \rightarrow V$ be given, we define $\mu(f)$, the minimal models of f :

$$\mu(f)(m) := \begin{cases} FALSE \quad iff & \forall \langle m, i\rangle \in \mathcal{X} \exists \langle m', i'\rangle \prec \langle m, i\rangle . f(m') \geq f(m) \\ \\ f(m) & otherwise \end{cases}$$

This generalizes the idea that only models of ϕ can destroy models of ϕ.

Obviously, for all $v \in V$, $v \neq FALSE$, $\{m : \mu(f)(m) = v\} \subseteq \{m : f(m) = v\}$.

A structure is called smooth iff for all f_ϕ and for all $\langle m, i\rangle$ such that there is $\langle m', i'\rangle \prec \langle m, i\rangle$ with $f_\phi(m') \geq f_\phi(m)$, there is $\langle m'', i''\rangle \prec \langle m, i\rangle$ with $f_\phi(m'') \geq f_\phi(m)$, and no $\langle n, j\rangle \prec \langle m'', i''\rangle$ with $f_\phi(n) \geq f_\phi(m'')$.

A structure will be called definablity preserving iff for all f_ϕ $\mu(f_\phi)$ is again the f_ψ for some ψ.

Definition 2.3.7

With these ideas, we can also define minimizing initial segments for many-valued structures in a straightforward way:

F is a MISE with respect to G iff

(0) $F \leq G$.

(1) if $F(x) \neq 0$, $y \prec x$, $G(x) \leq G(y)$, then $F(y) = G(y)$ (downward closure),

and

(2) if $G(x) \neq 0$, $F(x) = 0$, then there is y with $y \prec x$, $G(x) \leq G(y)$, $F(y) = G(y)$.

We turn to representation questions.

Example 2.3.2

This example shows that a suitable choice of truth values can destroy coherence, as it is present in 2-valued preferential structures.

We want essentially $y \prec x$ in A, $A \subseteq B$, but $y \not\prec x$ in B.

The solution will be to make

$F_B(y) < F_B(x)$, but $F_A(y) \geq F_A(x)$, and $F_A \leq F_B$, e.g., we set: $F_B(x) = 3$, $F_B(y) = 2$, $F_A(x) = F_A(y) = 2$.

This example leads to the following small representation result:

Fact 2.3.8

Let U be the universe we work in, let $\mu : \mathcal{P}(U) \to \mathcal{P}(U)$ be a function such that

(1) $\mu(X) \subseteq X$,

(2) there is no singleton $X = \{x\}$ with $\mu(X) = \emptyset$.

Then there is a many-valued preferential structure $\mathcal{X}$ which represents μ. Note that no coherence conditions are necessary.

Proof

Let 0 be the smallest truth value, and $\forall x \in U$ x also be a truth value, where for all $x \neq y$, $x, y \in U$, $x \perp y$ (x, y as truth values are incomparable). Take as preference relation $x \prec y$ for all $x, y \in U$, $x \neq y$.

Choose $X \subseteq U$. Define $F_X(x) := 0$ iff $x \notin \mu(X)$, and $F_X(x) := x$ iff $x \in \mu(X)$. Then all relations $x \prec y$ are effective for $y \notin \mu(X)$, as then $F_X(y) \leq F_X(x)$, so y will not be minimal. If $y \in \mu(X)$, then there is no $x \neq y$, $F_X(y) \leq F_X(x)$. $\Box$

Above Fact 2.3.8 (page 68) largely solves the problem of finding a preferential representation for arbitrary choice functions by many valued structures.

But one might ask different questions in this context, e.g.: Suppose we have a family of pairs $\langle F, \mu F \rangle$ of functions giving truth values to all $x \in U$. Suppose $\forall x \in U. \mu F(x) \leq F(x)$, in short $\mu F \leq F$. Suppose for simplicity that we have a minimal element 0 of truth values, with the meaning $F(x) = 0$ iff "$x \notin F$" (read as a set), so we will not consider x with $F(x) = 0$. Suppose further that $\mu F(x) = 0$ or $\mu F(x) = F(x)$. Then, what are the conditions on the family of $\langle F, \mu F \rangle$ such that we can represent them by a many-valued preferential structure? The answer is not as trivial as the one to the choice function representation problem above.

Consider the following

Example 2.3.3

Consider F, G with $F \leq G$, $F(x) \neq 0$, $\mu F(x) = 0$, $\mu F(y) \neq 0$, $F(x) \leq F(y)$. In this case, a relation $y \prec x$ is effective for F. Suppose now that also $G(x) \leq G(y)$, then $y \prec x$ is also effective for G. We may say roughly: If not only $F \leq G$, but for x, y such that $F(x), F(y) \neq 0$, also for the "derivatives" F' and G' $F' \leq G'$ holds in the sense that $F(x) \leq F(y) \Rightarrow G(x) \leq G(y)$, then F and G must have the same coherence properties as the 2-valued choice functions in order to be preferentially representable - as any relation effective for F will also be effective for G.

This may lead us to consider the following brutal solution:

We have a global truth value relation in the sense that for all F, G $F(x) \leq F(y)$ iff $G(x) \leq G(y)$ - apart from cases where, e.g., $F(x) = 0$, as "$x \notin F$". In this case, they behave just like normal 2-valued structures, but we could now just as well simply omit any relations $x \prec y$ when $F(y) \not\leq F(x)$ (equivalently, for any other G). So this leads us nowhere interesting.

We will leave the problem for further research, and only add a few rudimentary remarks:

(1) We may introduce new operators in order to be able to speak about the situation:

(a) $m \models O_F\phi$ iff for all m' such that $m' \models \phi$, $F(m) \leq F(m')$,

(b) $\models O_F(\phi, \psi)$ iff for all m such that $m \models \phi$ and all m' such that $m' \models \psi$, $F(m) \leq F(m')$.

These expressions are, of course, still semi-classical, and we can replace $\models$ by a certain threshold, or consider only m' such that $F_\phi(m') \geq F_\phi(m)$.

Then, given sufficient definability power, we can express that all models "in" μF have truth values at least as good as those in $F - \mu F := F \cap \boldsymbol{C}\mu F$, $\models O_F(F - \mu F, \mu F)$, and use this to formulate a coherence condition, like: $O_F(F - \mu F, \mu F) \Rightarrow O_G(F - \mu F, \mu F)$.

(2) To do some set theory, we will assume that the set of truth values is a complete Boolean algebra, with symbols $\wedge$ (or $\bigwedge$ for many arguments) for infimum, likewise $\vee$ and $\bigvee$ for supremum, unary - for complement, binary $a - b$ for $a \wedge -$b, 0 and 1. For functions F, G, etc with values in the set of truth values, we define $\wedge, \vee$ etc. argumentwise, e.g., $\bigwedge F_i$ is defined by $(\bigwedge F_i)(x) := \bigwedge(F_i(x))$ etc.

(3) As an illustration, and for no other purposes, we look at some cases of the crucial Fact 3.3.1 in [Sch04] for representation by smooth structures, which we repeat now here for easier reference, together with its proof:

Fact 2.3.9

Let A, U, U', Y and all A_i be in $\mathcal{Y}$.

$(\mu \subseteq)$ and (μPR) entail:

(1) $A = \bigcup\{A_i : i \in I\} \rightarrow \mu(A) \subseteq \bigcup\{\mu(A_i) : i \in I\}$,

(2) $U \subseteq H(U)$, and $U \subseteq U' \rightarrow H(U) \subseteq H(U')$,

(3) $\mu(U \cup Y) - H(U) \subseteq \mu(Y)$.

$(\mu \subseteq)$, (μPR), (μCUM) entail:

(4) $U \subseteq A$, $\mu(A) \subseteq H(U) \rightarrow \mu(A) \subseteq U$,

(5) $\mu(Y) \subseteq H(U) \rightarrow Y \subseteq H(U)$ and $\mu(U \cup Y) = \mu(U)$,

(6) $x \in \mu(U)$, $x \in Y - \mu(Y) \rightarrow Y \not\subseteq H(U)$,

(7) $Y \not\subseteq H(U) \rightarrow \mu(U \cup Y) \not\subseteq H(U)$.

Proof:

(1) $\mu(A) \cap A_j \subseteq \mu(A_j) \subseteq \bigcup \mu(A_i)$, so by $\mu(A) \subseteq A = \bigcup A_i$ $\mu(A) \subseteq \bigcup \mu(A_i)$.

(2) trivial.

(3) $\mu(U \cup Y) - H(U) \subseteq_{(2)} \mu(U \cup Y) - U \subseteq_{(\mu\subseteq)} \mu(U \cup Y) \cap Y \subseteq_{(\mu PR)} \mu(Y)$.

(4) $\mu(A) = \bigcup\{\mu(A) \cap X : \mu(X) \subseteq U\} \subseteq_{(\mu PR')} \bigcup\{\mu(A \cap X) : \mu(X) \subseteq U\}$. But if $\mu(X) \subseteq U \subseteq A$, then by $\mu(X) \subseteq X$, $\mu(X) \subseteq A \cap X \subseteq X \rightarrow_{(\mu CUM)} \mu(A \cap X) = \mu(X) \subseteq U$, so $\mu(A) \subseteq U$.

(5) Let $\mu(Y) \subseteq H(U)$, then by $\mu(U) \subseteq H(U)$ and (1) $\mu(U \cup Y) \subseteq \mu(U) \cup \mu(Y) \subseteq H(U)$, so by (4) $\mu(U \cup Y) \subseteq U$ and $U \cup Y \subseteq H(U)$. Moreover, $\mu(U \cup Y) \subseteq U \subseteq U \cup Y \rightarrow_{(\mu CUM)} \mu(U \cup Y) = \mu(U)$.

(6) If not, $Y \subseteq H(U)$, so $\mu(Y) \subseteq H(U)$, so $\mu(U \cup Y) = \mu(U)$ by (5), but $x \in Y - \mu(Y) \rightarrow_{(\mu PR)} x \notin \mu(U \cup Y) = \mu(U)$, *contradiction.*

(7) $\mu(U \cup Y) \subseteq H(U) \rightarrow_{(5)} U \cup Y \subseteq H(U)$. $\square$

We translate some properties and arguments:

- (1)
  (a) $F_i \leq F \Rightarrow \mu F \wedge F_i \leq \mu F_i$ for all i by (μPR) (but recall that (μPR) will not always hold, see Example 2.3.2 (page 68))
  (b) $\mu F \leq F \leq \bigvee_i F_i$.
  Thus $\mu F =$ (by *b*) $\mu F \wedge \bigvee_i F_i =$ (distributivity) $\bigvee_i(\mu F \wedge F_i) \leq$ (by a) $\bigvee_i \mu F_i$.
- (3)
  We first need an analogue to $X \subseteq Y \cup Z \Rightarrow X - Y \subseteq Z$:
  (a) $F_X \leq F_Y \vee F_Z \Rightarrow F_X - F_Y \leq F_Z$. Proof: $F_X - F_Y = F_X \wedge \boldsymbol{C}F_Y \leq$ (prerequisite) $(F_Y \wedge \boldsymbol{C}F_Y) \vee (F_Z \wedge \boldsymbol{C}F_Y) = 0 \vee (F_Z \wedge \boldsymbol{C}F_Y) \leq F_Z$.
  We then need
  (b) $F_X \leq F_{X'} \Rightarrow F_Y - F_{X'} \leq F_Y - F_X$. Proof: $F_X \leq F_{X'} \Rightarrow \boldsymbol{C}F_{X'} \leq \boldsymbol{C}F_X$, so $F_Y \cap \boldsymbol{C}F_{X'} \leq F_Y \cap \boldsymbol{C}F_X$.
  Thus, $\mu f_{U\cup Y} - f_{H(U)} \leq$ (by (2) and (b)) $\mu f_{U\cup Y} - f_U \leq$ (by $(\mu \subseteq)$, (a)) $\mu f_{U\cup Y} \wedge f_Y \leq \mu f_Y$ by (μPR).
- (6)
  $F_Y \leq F_{H(U)} \Rightarrow \mu F_Y \leq F_{H(U)} \Rightarrow \mu F_{U\cup Y} = \mu(F_U \vee F_Y) = \mu F_U$ by (5). $\mu F_{U\cup Y} \wedge F_Y = \mu(F_U \vee F_Y) \wedge F_Y \leq \mu F_Y$ by (μPR), so $\mu F_U \wedge F_Y = \mu F_{U\cup Y} \wedge F_Y \leq \mu F_Y$. Thus $\mu F_U \wedge F_Y \wedge \boldsymbol{C}\mu F_Y = \emptyset$, contradicting the prerequisite.

2.4 IBRS and higher preferential structures

2.4.1 General IBRS

We first define IBRS:

Definition 2.4.1

(1) An *information bearing binary relation frame IBR*, has the form $(S, \Re)$, where S is a non-empty set and $\Re$ is a subset of S_ω, where S_ω is defined by induction as follows:

(1.1) $S_0 := S$

(1.2) $S_{n+1} := S_n \cup (S_n \times S_n)$.

(1.3) $S_\omega = \bigcup \{S_n : n \in \omega\}$

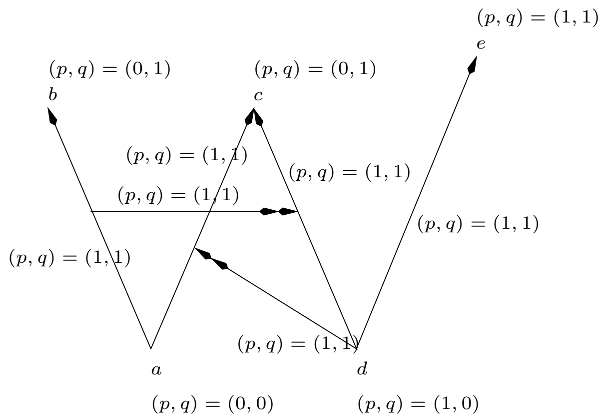
We call elements from S *points* or *nodes*, and elements from $\Re$ *arrows*. Given $(S, \Re)$, we also set $\boldsymbol{P}((S, \Re)) := S$, and $\boldsymbol{A}((S, \Re)) := \Re$.

If α is an arrow, the origin and destination of α are defined as usual, and we write $\alpha : x \to y$ when x is the origin, and y the destination of the arrow α. We also write $o(\alpha)$ and $d(\alpha)$ for the origin and destination of α.

(2) Let Q be a set of atoms, and $\boldsymbol{L}$ be a set of labels (usually $\{0,1\}$ or $[0,1]$). An *information assignment* h on $(S, \Re)$ is a function $h : Q \times \Re \to \boldsymbol{L}$.

(3) An *information bearing system IBRS*, has the form $(S, \Re, h, Q, \boldsymbol{L})$, where S, $\Re$, h, Q, $\boldsymbol{L}$ are as above.

See Diagram 2.4.1 (page 71) for an illustration.



A simple example of an information bearing system.

Diagram 2.4.1

We have here:

$$S = \{a, b, c, d, e\}.$$
$$\Re = S \cup \{(a, b), (a, c), (d, c), (d, e)\} \cup \{((a, b), (d, c)), (d, (a, c))\}.$$
$$Q = \{p, q\}$$

The values of h for p and q are as indicated in the figure. For example $h(p, (d, (a, c))) = 1$.

Comment 2.4.1

The elements in Figure Diagram 2.4.1 (page 71) can be interpreted in many ways, depending on the area of application.

(1) The points in S can be interpreted as possible worlds, or as nodes in an argumentation network or nodes in a neural net or states, etc.

(2) The direct arrows from nodes to nodes can be interpreted as accessibility relation, attack or support arrows in an argumentation networks, connection in a neural nets, a preferential ordering in a nonmonotonic model, etc.

(3) The labels on the nodes and arrows can be interpreted as fuzzy values in the accessibility relation or weights in the neural net or strength of arguments and their attack in argumentation nets, or distances in a counterfactual model, etc.

(4) The double arrows can be interpreted as feedback loops to nodes or to connections, or as reactive links changing the system which are activated as we pass between the nodes.

2.4.2 Higher preferential structures

2.4.2.1 Introduction

We turn to the special case of higher preferential structures, give the definitions and some results.

2.4.2.2 Definitions and facts for basic structures

Definition 2.4.2

An IBR is called a $generalized\ preferential\ structure$ iff the origins of all arrows are points. We will usually write x, y etc. for points, α, β etc. for arrows.

Definition 2.4.3

Consider a generalized preferential structure $\mathcal{X}$.

(1) *Level n arrow* :

Definition by upward induction.

If $\alpha : x \to y$, x, y are points, then α is a level 1 arrow.

If $\alpha : x \to \beta$, x is a point, β a level n arrow, then α is a level $n+1$ arrow. ($o(\alpha)$ is the origin, $d(\alpha)$ is the destination of α.)

$\lambda(\alpha)$ will denote the level of α.

(2) *Level n structure* :

$\mathcal{X}$ is a level n structure iff all arrows in $\mathcal{X}$ are at most level n arrows.

We consider here only structures of some arbitrary but finite level n.

(3) We define for an arrow α by induction $O(\alpha)$ and $D(\alpha)$.

If $\lambda(\alpha) = 1$, then $O(\alpha) := \{o(\alpha)\}$, $D(\alpha) := \{d(\alpha)\}$.

If $\alpha : x \to \beta$, then $D(\alpha) := D(\beta)$, and $O(\alpha) := \{x\} \cup O(\beta)$.

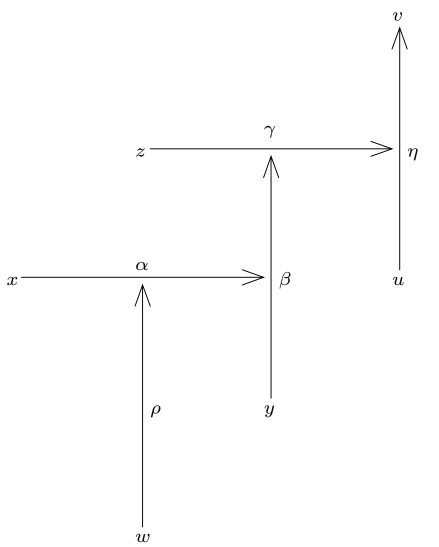
Thus, for example, if $\alpha : x \to y$, $\beta : z \to \alpha$, then $O(\beta) := \{x, z\}$, $D(\beta) = \{y\}$. Consider also the arrow $\beta := \langle \beta', l' \rangle$ in Diagram 2.4.4 (page 80). There, $D(\beta) = \{\langle x, i \rangle\}$, $O(\beta) = \{\langle z', m' \rangle, \langle y, j \rangle\}$.

Example 2.4.1

Let $\lambda(\alpha)$ be finite. α may be of the form $\alpha : x \to \beta$, $\beta : y \to \gamma$, $\gamma : z \to \eta$, $\eta : u \to v$. $D(\alpha) = \{v\}$, the last destination in the construction, so always a point. $O(\alpha)$ is $\{x, y, z, u\}$, the set of all origins of these arrows, a set of points. We do *not* consider any other arrows pointing to or going from elements of this construction.

See Diagram 2.4.2 (page 73).

Diagram 2.4.2



$D(x) = \{v\},\ O(x) = \{x, y, z, u\}$

Comment 2.4.2

A counterargument to α is NOT an argument for $\neg\alpha$ (this is asking for too much), but just showing one case where $\neg\alpha$ holds. In preferential structures, an argument for α is a set of level 1 arrows, eliminating $\neg\alpha-$models. A counterargument is one level 2 arrow, attacking one such level 1 arrow.

Of course, when we have copies, we may need many successful attacks, on all copies, to achieve the goal. As we may have copies of level 1 arrows, we may need many level 2 arrows to destroy them all.

We will not consider here diagrams with arbitrarily high levels. One reason is that diagrams like the following will have an unclear meaning:

Example 2.4.2

$\langle\alpha, 1\rangle : x \to y,$

$\langle\alpha, n+1\rangle : x \to \langle\alpha, n\rangle\ (n \in \omega).$

Is $y \in \mu(X)$?

Definition 2.4.4

Let $\mathcal{X}$ be a generalized preferential structure of (finite) level $n.$

We define (by downward induction):

(1) $Valid\ X-to-Y\ arrow:$

Let $X, Y \subseteq \boldsymbol{P}(\mathcal{X}).$

$\alpha \in \boldsymbol{A}(\mathcal{X})$ is a $valid\ X-to-Y\ arrow$ iff

(1.1) $O(\alpha) \subseteq X$, $D(\alpha) \subseteq Y$,

(1.2) $\forall\beta : x' \to \alpha.(x' \in X \Rightarrow \exists\gamma : x'' \to \beta.(\gamma$ is a valid $X-to-Y$ arrow)).

We will also say that α is a $valid\ arrow$ in X, or just $valid$ in X, iff α is a valid $X-to-X$ arrow.

(2) $Valid\ X \Rightarrow Y\ arrow:$

Let $X \subseteq Y \subseteq \boldsymbol{P}(\mathcal{X}).$

$\alpha \in \boldsymbol{A}(\mathcal{X})$ is a $valid\ X \Rightarrow Y\ arrow$ iff

(2.1) $o(\alpha) \in X$, $O(\alpha) \subseteq Y$, $D(\alpha) \subseteq Y$,

(2.2) $\forall\beta : x' \to \alpha.(x' \in Y \Rightarrow \exists\gamma : x'' \to \beta.(\gamma$ is a valid $X \Rightarrow Y$ arrow)).

Thus, any attack β from Y against α has to be countered by a valid attack on β.

(Note that in particular $o(\gamma) \in X$, and that $o(\beta)$ need not be in X, but can be in the bigger Y.)

Remark 2.4.1

Note that, in the definition of valid $X-to-Y$ arrow, X and Y need not be related, but in the definition of valid $X \Rightarrow Y$ arrow, $X \subseteq Y$.

Let us assume now that $X \subseteq Y$, and look at the remaining differences.

In both cases, $D(\alpha) \subseteq Y$.

In the $X - to - Y$ case, $O(\alpha) \subseteq X$, and attacks from X are countered.

In the $X \Rightarrow Y$ case, $o(\alpha) \in X$, $O(\alpha) \subseteq Y$, and attacks from Y are countered.

So the first condition is stronger in the $X - to - Y$ case, the second in the $X \Rightarrow Y$ case.

Example 2.4.3

(1) Consider the arrow $\beta := \langle \beta', l' \rangle$ in Diagram 2.4.4 (page 80). $D(\beta) = \{\langle x, i \rangle\}$, $O(\beta) = \{\langle z', m' \rangle, \langle y, j \rangle\}$, and the only arrow attacking β originates outside X, so β is a valid $X - to - \mu(X)$ arrow.

(2) Consider the arrows $\langle \alpha', k' \rangle$ and $\langle \gamma', n' \rangle$ in Diagram 2.4.5 (page 80). Both are valid $\mu(X) \Rightarrow X$ arrows.

Example 2.4.4

See Diagram 2.4.3 (page 76).

Let $X \subseteq Y$.

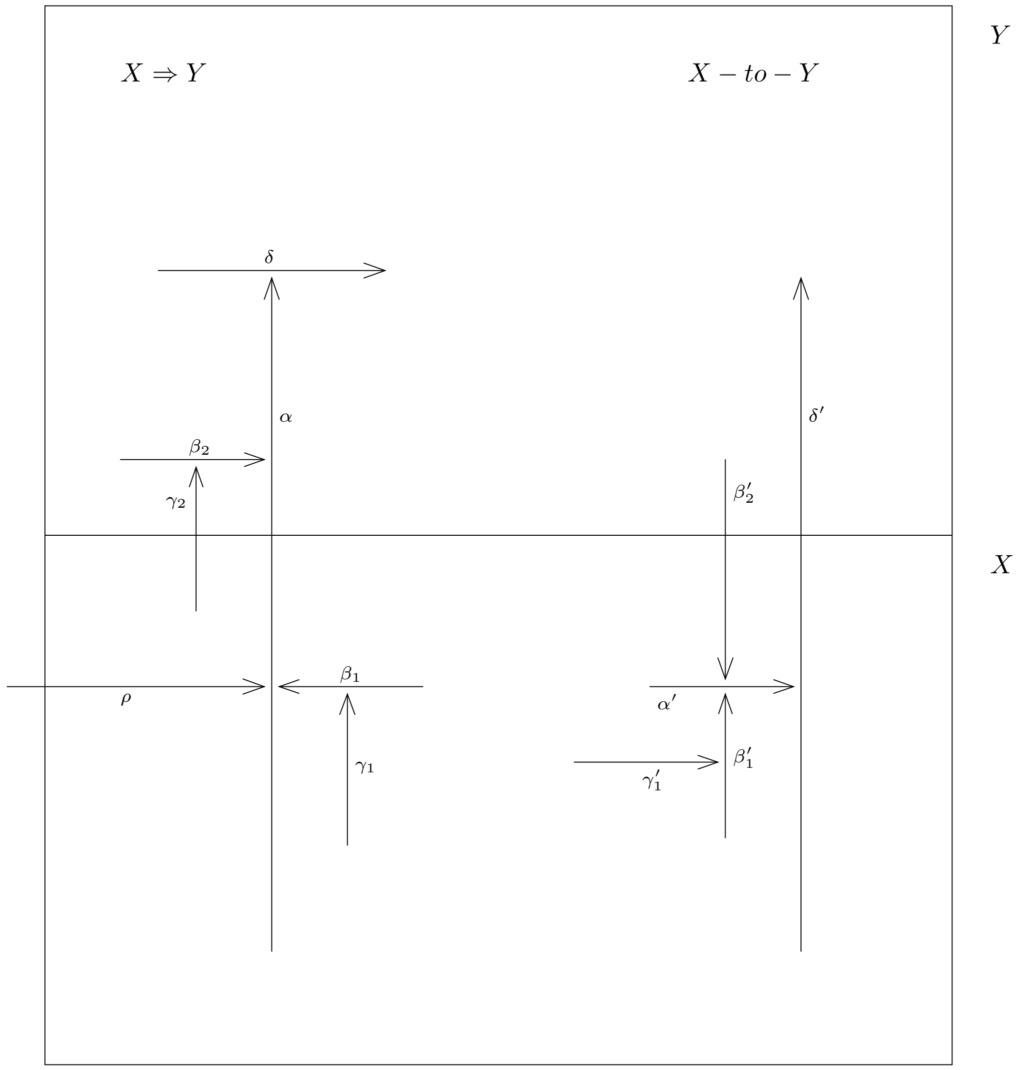
Consider the left hand part of the diagram.

The fact that δ originates in Y, but not in X makes that α is not a valid $X - to - Y$ arrow, as the condition $O(\alpha) \subseteq X$ is violated. To be a valid $X \Rightarrow Y$ arrow, we have to show that all attacks on α originating from Y (not only from X) are be countered by valid $X \Rightarrow Y$ arrows. This holds, as β_1 is countered by γ_1, β_2 by γ_2. All possible attacks on α, γ_1, γ_2 from outside Y, like ρ, need not be considered.

Consider the right hand part of the diagram.

The fact that there is no valid counterargument to β'_2 makes that α' is not a valid $X \Rightarrow Y$ arrow. It is a valid $X - to - Y$ arrow, as counterarguments to α' like β'_2, which do not originate in X, are not considered. The counterargument β'_1 is considered, but it is destroyed by the valid $X - to - Y$ arrow γ'_1.

Diagram 2.4.3



Fact 2.4.2

(1) If α is a valid $X \Rightarrow Y$ arrow, then α is a valid $Y - to - Y$ arrow.

(2) If $X \subseteq X' \subseteq Y' \subseteq Y \subseteq \boldsymbol{P}(\mathcal{X})$ and $\alpha \in \boldsymbol{A}(\mathcal{X})$ is a valid $X \Rightarrow Y$ arrow, and $O(\alpha) \subseteq Y'$, $D(\alpha) \subseteq Y'$, then α is a valid $X' \Rightarrow Y'$ arrow.

Proof

Let α be a valid $X \Rightarrow Y$ arrow. We show (1) and (2) together by downward induction (both are trivial).

By prerequisite $o(\alpha) \in X \subseteq X'$, $O(\alpha) \subseteq Y' \subseteq Y$, $D(\alpha) \subseteq Y' \subseteq Y$.

Case 1: $\lambda(\alpha) = n$. So α is a valid $X' \Rightarrow Y'$ arrow, and a valid $Y - to - Y$ arrow.

Case 2: $\lambda(\alpha) = n - 1$. So there is no $\beta : x' \to \alpha$, $y \in Y$, so α is a valid $Y - to - Y$ arrow. By $Y' \subseteq Y$ α is a valid $X' \Rightarrow Y'$ arrow.

Case 3: Let the result be shown down to m, $n > m > 1$, let $\lambda(\alpha) = m - 1$. So $\forall \beta : x' \to \alpha(x' \in Y \Rightarrow \exists \gamma : x'' \to \beta(x'' \in X$ and γ is a valid $X \Rightarrow Y$ arrow$))$. By induction hypothesis γ is a valid $Y - to - Y$ arrow, and a valid $X' \Rightarrow Y'$ arrow. So α is a valid $Y - to - Y$ arrow, and by $Y' \subseteq Y$, α is a valid $X' \Rightarrow Y'$ arrow.

$\square$

Definition 2.4.5

Let $\mathcal{X}$ be a generalized preferential structure of level n, $X \subseteq \boldsymbol{P}(\mathcal{X})$.

$\mu(X) := \{x \in X : \exists \langle x, i \rangle . \neg \exists$ valid $X - to - X$ arrow $\alpha : x' \to \langle x, i \rangle\}$.

2.4.2.3 Definitions and facts for totally and essentially smooth structures

Comment 2.4.3

The purpose of smoothness is to guarantee cumulativity. Smoothness achieves Cumulativity by mirroring all information present in X also in $\mu(X)$. Closer inspection shows that smoothness does more than necessary. This is visible when there are copies (or, equivalently, non-injective labelling functions). Suppose we have two copies of $x \in X$, $\langle x, i \rangle$ and $\langle x, i' \rangle$, and there is $y \in X$, $\alpha : \langle y, j \rangle \to \langle x, i \rangle$, but there is no $\alpha' : \langle y', j' \rangle \to \langle x, i' \rangle$, $y' \in X$. Then $\alpha : \langle y, j \rangle \to \langle x, i \rangle$ is irrelevant, as $x \in \mu(X)$ anyhow. So mirroring $\alpha : \langle y, j \rangle \to \langle x, i \rangle$ in $\mu(X)$ is not necessary, i.e., it is not necessary to have some $\alpha' : \langle y', j' \rangle \to \langle x, i \rangle$, $y' \in \mu(X)$.

On the other hand, Example 2.4.6 (page 82) shows that, if we want smooth structures to correspond to the property (μCUM), we need at least some valid arrows from $\mu(X)$ also for higher level arrows. This "some" is made precise (essentially) in Definition 2.4.6 (page 79).

From a more philosophical point of view, when we see the (inverted) arrows of preferential structures as attacks on non-minimal elements, then we should see smooth structures as always having attacks also from valid (minimal) elements. So, in general structures, also attacks from non-valid elements are valid; in smooth structures we always also have attacks from valid elements.

The analogue to usual smooth structures, on level 2, is then that any successfully attacked level 1 arrow is also attacked from a minimal point.

Definition 2.4.6

Let $\mathcal{X}$ be a generalized preferential structure.

$X \sqsubseteq X'$ iff

(1) $X \subseteq X' \subseteq \boldsymbol{P}(\mathcal{X})$,

(2) $\forall x \in X' - X\ \forall \langle x, i\rangle\ \exists \alpha : x' \to \langle x, i\rangle (\alpha$ is a valid $X \Rightarrow X'$ arrow),

(3) $\forall x \in X\ \exists \langle x, i\rangle$

$(\forall \alpha : x' \to \langle x, i\rangle (x' \in X' \Rightarrow \exists \beta : x'' \to \alpha.(\beta$ is a valid $X \Rightarrow X'$ arrow$)))$.

Note that (3) is not simply the negation of (2):

Consider a level 1 structure. Thus all level 1 arrows are valid, but the source of the arrows must not be neglected.

(2) reads now: $\forall x \in X' - X\ \forall \langle x, i\rangle\ \exists \alpha : x' \to \langle x, i\rangle . x' \in X$

(3) reads: $\forall x \in X\ \exists \langle x, i\rangle\ \neg \exists \alpha : x' \to \langle x, i\rangle . x' \in X'$

This is intended: intuitively, read $X = \mu(X')$, and minimal elements must not be attacked at all, but non-minimals must be attacked from X - which is a modified version of smoothness. More precisely: non-minimal elements (i.e., from $X'-$X) have to be validly attacked from X, minimal elements must not be validly attacked at all from X' (only perhaps from the outside).

Remark 2.4.3

We note the special case of Definition 2.4.6 (page 79) for level 3 structures. We also write it immediately for the intended case $\mu(X) \sqsubseteq X$, and explicitly with copies.

$x \in \mu(X)$ iff

(1) $\exists \langle x, i\rangle \forall \langle \alpha, k\rangle : \langle y, j\rangle \to \langle x, i\rangle$

$(y \in X \Rightarrow \exists \langle \beta', l'\rangle : \langle z', m'\rangle \to \langle \alpha, k\rangle .$

$(z' \in \mu(X) \wedge \neg \exists \langle \gamma', n'\rangle : \langle u', p'\rangle \to \langle \beta', l'\rangle . u' \in X))$

See Diagram 2.4.4 (page 80).

$x \in X - \mu(X)$ iff

(2) $\forall \langle x, i\rangle \exists \langle \alpha', k'\rangle : \langle y', j'\rangle \to \langle x, i\rangle$

$(y' \in \mu(X) \wedge$

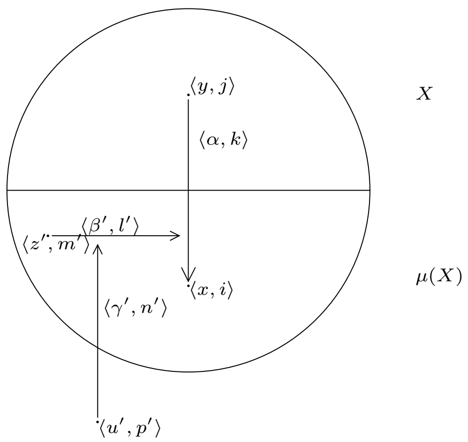
(a) $\neg\exists \langle \beta', l'\rangle : \langle z', m'\rangle \to \langle \alpha', k'\rangle . z' \in X$

or

(b) $\forall \langle \beta', l'\rangle : \langle z', m'\rangle \to \langle \alpha', k'\rangle$

$(z' \in X \Rightarrow \exists \langle \gamma', n'\rangle : \langle u', p'\rangle \to \langle \beta', l'\rangle . u' \in \mu(X)))$

See Diagram 2.4.5 (page 80).



Case 3-1-2

Diagram 2.4.4

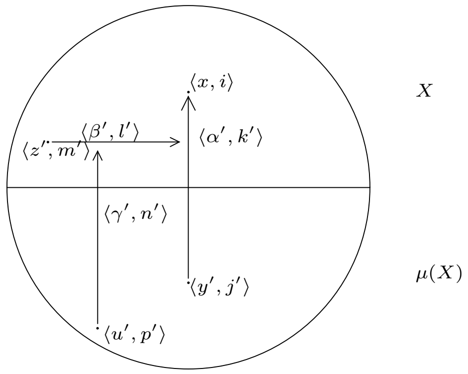


Diagram 2.4.5 **Case 3-2**

Fact 2.4.4

(1) If $X \sqsubseteq X'$, then $X = \mu(X')$,

(2) $X \sqsubseteq X'$, $X \subseteq X'' \subseteq X' \Rightarrow X \sqsubseteq X''$. (This corresponds to (μCUM) .)

(3) $X \sqsubseteq X'$, $X \subseteq Y'$, $Y \sqsubseteq Y'$, $Y \subseteq X' \Rightarrow X = Y$. (This corresponds to $(\mu \subseteq\supseteq)$.)

Proof

(1) Trivial by Fact 2.4.2 (page 77) (1).

(2)

We have to show

(a) $\forall x \in X'' - X\ \forall \langle x, i\rangle\ \exists \alpha : x' \to \langle x, i\rangle(\alpha$ is a valid $X \Rightarrow X''$ arrow), and

(b) $\forall x \in X\ \exists \langle x, i\rangle\ (\forall \alpha : x' \to \langle x, i\rangle(x' \in X'' \Rightarrow \exists \beta : x'' \to \alpha.(\beta$ is a valid $X \Rightarrow X''$ arrow))).

Both follow from the corresponding condition for $X \Rightarrow X'$, the restriction of the universal quantifier, and Fact 2.4.2 (page 77) (2).

(3)

Let $x \in X-$Y.

(a) By $x \in X \sqsubseteq X'$, $\exists \langle x, i\rangle$ s.t. $(\forall \alpha : x' \to \langle x, i\rangle(x' \in X' \Rightarrow \exists \beta : x'' \to \alpha.(\beta$ is a valid $X \Rightarrow X'$ arrow))).

(b) By $x \notin Y \sqsubseteq \exists \alpha_1 : x' \to \langle x, i\rangle$ α_1 is a valid $Y \Rightarrow Y'$ arrow, in particular $x' \in Y \subseteq X'$. Moreover, $\lambda(\alpha_1) = 1$.

So by (a) $\exists \beta_2 : x'' \to \alpha_1.(\beta_2$ is a valid $X \Rightarrow X'$ arrow), in particular $x'' \in X \subseteq Y'$, moreover $\lambda(\beta_2) = 2$.

It follows by induction from the definition of valid $A \Rightarrow B$ arrows that

$\forall n \exists \alpha_{2m+1}$, $\lambda(\alpha_{2m+1}) = 2m+1$, α_{2m+1} a valid $Y \Rightarrow Y'$ arrow and

$\forall n \exists \beta_{2m+2}$, $\lambda(\beta_{2m+2}) = 2m+2$, β_{2m+2} a valid $X \Rightarrow X'$ arrow,

which is impossible, as $\mathcal{X}$ is a structure of finite level.

□

Definition 2.4.7

Let $\mathcal{X}$ be a generalized preferential structure, $X \subseteq \boldsymbol{P}(\mathcal{X})$.

$\mathcal{X}$ is called *totally smooth* for X iff

(1) $\forall \alpha : x \to y \in \boldsymbol{A}(\mathcal{X})(O(\alpha) \cup D(\alpha) \subseteq X \Rightarrow \exists \alpha' : x' \to y.x' \in \mu(X))$

(2) if α is valid, then there must also exist such α' which is valid.

(y a point or an arrow).

If $\mathcal{Y} \subseteq \boldsymbol{P}(\mathcal{X})$, then $\mathcal{X}$ is called $\mathcal{Y} - totally\ smooth$

iff for all $X \in \mathcal{Y}$ $\mathcal{X}$ is totally smooth for X.

Example 2.4.5

$X := \{\alpha : a \to b,\ \alpha' : b \to c,\ \alpha'' : a \to c,\ \beta : b \to \alpha'\}$ is not totally smooth,

$X := \{\alpha : a \to b,\ \alpha' : b \to c,\ \alpha'' : a \to c,\ \beta : b \to \alpha',\ \beta' : a \to \alpha'\}$ is totally smooth.

Example 2.4.6

Consider $\alpha' : a \to b$, $\alpha'' : b \to c$, $\alpha : a \to c$, $\beta : a \to \alpha$.

Then $\mu(\{a, b, c\}) = \{a\}$, $\mu(\{a, c\}) = \{a, c\}$. Thus, (μCUM) does not hold in this structure. Note that there is no valid arrow from $\mu(\{a, b, c\})$ to c.

Definition 2.4.8

Let $\mathcal{X}$ be a generalized preferential structure, $X \subseteq \boldsymbol{P}(\mathcal{X})$.

$\mathcal{X}$ is called *essentially smooth* for X iff $\mu(X) \sqsubseteq X$. If $\mathcal{Y} \subseteq \boldsymbol{P}(\mathcal{X})$, then $\mathcal{X}$ is called $\mathcal{Y}-essentially$ *smooth*

iff for all $X \in \mathcal{Y}$ $\mu(X) \sqsubseteq X$.

2.4.2.4 Semantic representation results for higher preferential structures

Result on not necessarily smooth structures

We give a representation theorem, but will make it more general than for preferential structures only. For this purpose, we will introduce some definitions first.

Definition 2.4.9

Let $\eta, \rho : \mathcal{Y} \to \mathcal{P}(U)$.

(1) If $\mathcal{X}$ is a simple structure:

$\mathcal{X}$ is called an *attacking structure* relative to η representing ρ iff

$\rho(X) = \{x \in \eta(X) :$ there is no valid $X - to - \eta(X)$ arrow $\alpha : x' \to x\}$

for all $X \in \mathcal{Y}$.

(2) If $\mathcal{X}$ is a structure with copies:

$\mathcal{X}$ is called an *attacking structure* relative to η representing ρ iff

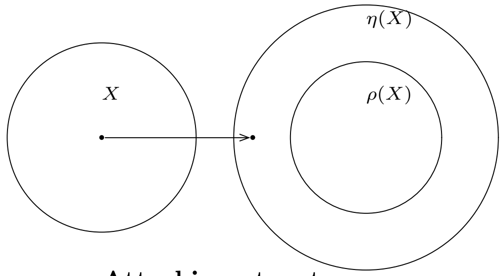
$\rho(X) = \{x \in \eta(X) :$ there is $\langle x, i \rangle$ and no valid $X - to - \eta(X)$ arrow $\alpha : \langle x', i' \rangle \to \langle x, i \rangle\}$

for all $X \in \mathcal{Y}$.

Obviously, in those cases $\rho(X) \subseteq \eta(X)$ for all $X \in \mathcal{Y}$.

Thus, $\mathcal{X}$ is a preferential structure iff η is the identity.

See Diagram 2.4.6 (page 83)



Attacking structure

Diagram 2.4.6

The following result shows that we can obtain (almost) anything with level 2 structures.

Proposition 2.4.5

Let $\eta, \rho : \mathcal{Y} \to \mathcal{P}(U)$. Then there is an attacking level 2 structure relative to η representing ρ iff
(1) $\rho(X) \subseteq \eta(X)$ for all $X \in \mathcal{Y}$,
(2) $\rho(\emptyset) = \eta(\emptyset)$ if $\emptyset \in \mathcal{Y}$.

(2) is, of course, void for preferential structures.

Results on essential smoothness

Definition 2.4.10

Let $\mu : \mathcal{Y} \to \mathcal{P}(U)$ and $\mathcal{X}$ be given, let $\alpha : \langle y, j \rangle \to \langle x, i \rangle \in \mathcal{X}$.
Define
$\boldsymbol{O}(\alpha) := \{Y \in \mathcal{Y} : x \in Y - \mu(Y), y \in \mu(Y)\}$,
$\boldsymbol{D}(\alpha) := \{X \in \mathcal{Y} : x \in \mu(X), y \in X\}$,
$\Pi(\boldsymbol{O}, \alpha) := \Pi\{\mu(Y) : Y \in \boldsymbol{O}(\alpha)\}$,
$\Pi(\boldsymbol{D}, \alpha) := \Pi\{\mu(X) : X \in \boldsymbol{D}(\alpha)\}$.

Lemma 2.4.6

Let U be the universe, $\mu : \mathcal{Y} \to \mathcal{P}(U)$. Let μ satisfy $(\mu \subseteq) + (\mu \subseteq\supseteq)$.

Let $\mathcal{X}$ be a level 1 preferential structure, $\alpha : \langle y, j\rangle \to \langle x, i\rangle$, $\boldsymbol{O}(\alpha) \neq \emptyset$, $\boldsymbol{D}(\alpha) \neq \emptyset$.

We can modify $\mathcal{X}$ to a level 3 structure $\mathcal{X}'$ by introducing level 2 and level 3 arrows s.t. no copy of α is valid in any $X \in \boldsymbol{D}(\alpha)$, and in every $Y \in \boldsymbol{O}(\alpha)$ at least one copy of α is valid. (More precisely, we should write $\mathcal{X}' \upharpoonright X$ etc.)

Thus, in $\mathcal{X}'$,

(1) $\langle x, i\rangle$ will not be minimal in any $Y \in \boldsymbol{O}(\alpha)$,

(2) if α is the only arrow minimizing $\langle x, i\rangle$ in $X \in \boldsymbol{D}(\alpha)$, $\langle x, i\rangle$ will now be minimal in X.

The construction is made independently for all such arrows $\alpha \in \mathcal{X}$.

Proposition 2.4.7

Let U be the universe, $\mu : \mathcal{Y} \to \mathcal{P}(U)$.

Then any μ satisfying $(\mu \subseteq)$, $(\cap)$, (μCUM) (or, alternatively, $(\mu \subseteq)$ and $(\mu \subseteq\supseteq)$) can be represented by a level 3 essentially smooth structure.

2.4.2.5 Translation to logics

We turn to the translation to logics.

Proposition 2.4.8

Let $\mid\sim$ be a logic for $\mathcal{L}$. Set $T^{\mathcal{M}} := Th(\mu_{\mathcal{M}}(M(T)))$, where $\mathcal{M}$ is a generalized preferential structure, and $\mu_{\mathcal{M}}$ its choice function. Then

(1) there is a level 2 preferential structure $\mathcal{M}$ s.t. $\overline{\overline{T}} = T^{\mathcal{M}}$ iff (LLE) , (CCL) , (SC) hold for all $T, T' \subseteq \mathcal{L}$.

(2) there is a level 3 essentially smooth preferential structure $\mathcal{M}$ s.t. $\overline{\overline{T}} = T^{\mathcal{M}}$ iff (LLE), (CCL), (SC), $(\subseteq\supseteq)$ hold for all $T, T' \subseteq \mathcal{L}$.

2.4.2.6 Discussion of the limit version of higher preferential structures

In usual preferential structures, the definition of a MISE in X, a minimizing initial segment of X, was clear: It should minimize all other elements of X, and it should be downward closed. This is an intuitively clear generalization of the set of minimal elements. The minimal elements are the best elements, and the least minimal ones the worst. The set of minimal elements is the ideal, this set may be empty, but leaving aside some of the worst is an approximation. This is the aim, to define what holds in the limit, in a sufficiently good approximation of the ideal. A good approximation should not exclude better elements, so it has to be downward closed. (In addition, this property assures that the intersection of two MISE will not be empty, which is very important, as we do not want to conclude FALSE.) As a MISE will minimize all other elements, we assure that finally all bad elements may be left out. This is all so clear and simple, as the relation expresses a clear relation of quality. If $x \prec y \prec z$, then x is better than y, y is better than z, and adding transitivity is natural: x is better than z.

This intuition is not clear for higher preferential structures. If we read $x \prec y$ as "x attacks y", then $x \prec y \prec z$ may mean that x also attacks the attack $y \prec z$. Of course, we can argue, to express this idea, it suffices to explicitly add an attack from x to the attack $\alpha : y \prec z$ itself (not the starting point y), an arrow $\beta : x \to \alpha$. Thus, we can assume transitivity, if we do not want it, we just write this down using $\beta : x \to \alpha$. But then we also may have to add some destruction of the resulting arrow $\alpha' : x \to z$, this can be done, take $\beta' : x \to \alpha'$. This is already somewhat cumbersome. But sometimes we are interested mainly in *valid* (in some sense) arrows, and concatenating them again to a valid arrow will destroy the possibility of adding such an arrow β'. So we may still destroy α, but not α' any more, which seems an inconvenient.

The basic problem is that we have no clear intuition about the "quality" of points, in the way we had it for usual preferential structures - as described above. Asking for strong formal properties like transitivity of "valid" arrows is perhaps counterintuitive. Unfortunately, assuming transitivity was at the heart of a reasonable definition of a MISE (assuring among other things that finite AND holds: the intersection of two MISE is a MISE, see Fact 2.3.2 (page 63) (2)). Of course, we can put our qualms about intuition aside and assume sufficient transitivity. But we see immediately a new problem. One of the nice properties of MISE was to assure a unitary form of cumulativity, see Fact 2.3.4 (page 65), again related to transitivity: If $A \subseteq B$ is a MISE in B, $B \subseteq C$ a MISE in C, then $A \subseteq C$ is a MISE in C, see Fact 2.3.2 (page 63), (2a). In higher preferential structures, we would want to have (among other properties), that if $c \in C - B$, then there is a "valid" arrow α from some $b \in B$ to c, $\alpha : b \to c$. See Diagram 2.4.7 (page 88). Moreover, if $b \in B - A$, then there should be a "valid" arrow from some a to b, $\alpha' : a \to b$. Finally, we will want some "valid" arrow $\alpha'' : a \to c$. What does "valid" mean here. If it means that there is no (valid) attack against the arrow in the big set, i.e., no valid attack against α from C, no valid attack against α' from B, then it is unclear why there should not be any valid attack against α'' from C, as "A does not know about C, only about B". Thus, we have to restrict ourselves to consider attacks against α from B, against α' from A, and again against α'' from A.

In the Diagram 2.4.7 (page 88), α is attacked by β_0 from C, by β from B, but only the latter attack has to be countered by γ. The situation is similar for α'. By transitivity, there is now α''. The attack β_0'' need not be countered, but the attack β'', it is countered by γ''.

But why should an attack β_0'' be admitted without counterattack? The only reason seems to be that it originates from elements which are attacked themselves. So an attack against an element weakens attacks originating from this element, something we had doubts about.

It seems difficult to find a plausible solution without a clear intuition about strength. Thus, we limit ourselves to a very general and informal description, and leave the question open for further research. A MISE $A \subseteq B$ for a higher preferential structure should have the following vague properties:

(1) For all $b \in B - A$ there should be $a \in A$ and a valid arrow $\alpha : a \to b$.

(2) A should be downward closed in B : If there is a valid arrow $\alpha : b \to a$ to some $a \in A$, from some $b \in B$, then b should be in A. Or, in other words, if $b \in B-$A, then every arrow $\alpha : b \to a$ into A should be validly attacked.

The next section is an attempt to clarify the intuition - of one of us (Karl Schlechta). It seems that the authors have here somewhat diverging intuitions.

2.4.2.7 Ideas for the intuitive meaning of higher preferential structures

It seems we have to differentiate:

(1) relations like in preferential structures (seen as arrows)

(2) arrows of support

(3) arrows of attack

Examples:

(1) Consider $a \to b$, $c \to b$.

 (1.1) In preferential structures, this means that a and c are better than b. If we just have $a \to b$, this does not mean that b is now better than in the first situation.

 (1.2) If we see arrows as support, in the first situation, b is better than in the second situation.

 (1.3) But if we see arrows as attacks, in the second situation, there are less attacks against b, so b is better.

(2) Consider $a \to b \to c$.

 (2.1) In preferential structures, transitivity is reasonable, considering arrows as expressing quality.

 (2.2) So it is for arrows as support, but it seems reasonable to say that the support from a is less strong than the support from b, being indirect support.

 (2.3) In the third case, transitivity is not wanted, on the contrary. The attack of a on b attenuates the attack of b on c. Still, it seems reasonable to say that the attack does not disappear totally.

 (2.4) Suppose we add a supporting arrow $d \to c$.

 Then, if all are supports, the whole situation is better for c than the situation with only $d \to c$.

 If the arrows $a \to b \to c$ are attacks, then the whole situation is worse for c than the situation with only $d \to c$.

(3) Suppose we go upward from a given set X of points. So all $x \in X$ will have full credibility.

 (3.1) For preference relations, we get worse going upward.

(3.2) For support, the longer a support chain, the weaker the support it gives. Different chains add their support (we have to define different!).

(3.3) For attack, we alternate, but attacks get weaker with longer chains. Again, different attacks add up.

(4) Suppose we go downward from some point x in an infinite descending chain.

(4.1) In the preferential case, we just get better, without any optimum, this makes sense.

(4.2) In the support case, we have "support out of thin air", such chains should not be considered.

E.g., we have arguments that the moon is made from cheese: The arguments go like this: A_0 : We believe that there is a herd of at least 10^{10} cows hidden behind the moon.

A_1 : We believe that there is a herd of at least $10^{10}+1$ cows hidden behind the moon.

A_2 : We believe that there is a herd of at least $10^{10}+2$ cows hidden behind the moon.

Etc. Of course, A_{n+1} implies A_n, and A_0 makes the cheese hypothesis seem reasonable.

(4.3) In the attack case, we have "attack out of thin air", again, such chains should not be considered.

(5) Cycles:

(5.1) Preferences: the intuitively best interpretation is probably to see them as $\preceq$, and not $\prec$.

(5.2) For the other cases, it seems we have to distinguish whether we go into a cycle, or come out of it:

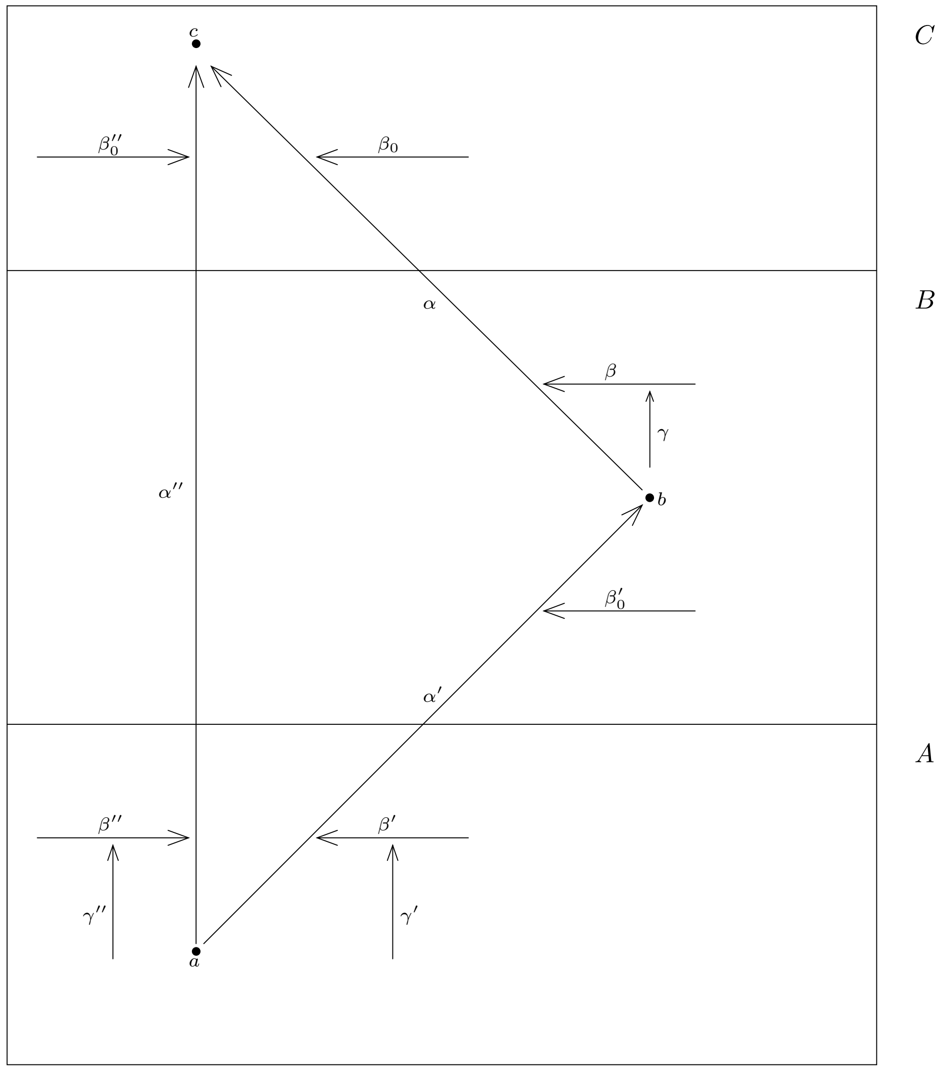
If we go into a cycle, we should stop after going around once, and treat the result as usual.

If we come out of a cycle (with no linear path going into it) we should consider it just as an infinite descending chain, i.e., neglect it.

(6) Attacks on attacks etc., some attenuation, like length of path, should be considered. This is important for the limit version of higher preferential structures.

(7) Do attacks and support only go against other arrows, or also against points?

Diagram 2.4.7



2.5 Theory revision

We give here the basic concepts and ideas of theory revision, first the AGM approach, and then distance based revision.

All material in this Section 2.5 (page 90) is due verbatim or in essence to AGM - AGM for Alchourron, Gardenfors, Makinson, see e.g., [AGM85].

Definition 2.5.1

We present in parallel the logical and the semantic (or purely algebraic) side. For the latter, we work in some fixed universe U, and the intuition is $U = M_{\mathcal{L}}$, $X = M(K)$, etc., so, e.g., $A \in K$ becomes $X \subseteq B$, etc.

(For reasons of readability, we omit most caveats about definability.)

$K_{\perp}$ will denote the inconsistent theory.

We consider two functions, - and $*$, taking a deductively closed theory and a formula as arguments, and returning a (deductively closed) theory on the logics side. The algebraic counterparts work on definable model sets. It is obvious that $(K-1)$, $(K*1)$, $(K-6)$, $(K*6)$ have vacuously true counterparts on the semantical side. Note that K (X) will never change, everything is relative to fixed K (X). $K*\phi$ is the result of revising K with ϕ. $K-\phi$ is the result of subtracting enough from K to be able to add $\neg\phi$ in a reasonable way, called contraction.

Moreover, let $\leq_K$ be a relation on the formulas relative to a deductively closed theory K on the formulas of $\mathcal{L}$, and $\leq_X$ a relation on $\mathcal{P}(U)$ or a suitable subset of $\mathcal{P}(U)$ relative to fixed X. When the context is clear, we simply write $\leq$. $\leq_K$ $(\leq_X)$ is called a relation of epistemic entrenchment for K (X).

Table 2.5 (page 94), "AGM theory revision", presents "rationality postulates" for contraction (-), rationality postulates revision $(*)$ and epistemic entrenchment. In AGM tradition, K will be a deductively closed theory, ϕ, ψ formulas. Accordingly, X will be the set of models of a theory, A, B the model sets of formulas.

In the further development, formulas ϕ etc. may sometimes also be full theories. As the transcription to this case is evident, we will not go into details.

Remark 2.5.1

(1) Note that $(X \mid 7)$ and $(X \mid 8)$ express a central condition for ranked structures: If we note $X \mid .$ by $f_X(.)$, we then have: $f_X(A) \cap B \neq \emptyset \Rightarrow f_X(A \cap B) = f_X(A) \cap B$.

(2) It is trivial to see that AGM revision cannot be defined by an individual distance (see Definition 2.5.3 (page 91)): Suppose $X \mid Y := \{y \in Y : \exists x_y \in X(\forall y' \in Y.d(x_y, y) \leq d(x_y, y'))\}$. Consider a, b, c. $\{a, b\} \mid \{b, c\} = \{b\}$ by $(X \mid 3)$ and $(X \mid 4)$, so $d(a, b) < d(a, c)$. But on the other hand $\{a, c\} \mid \{b, c\} = \{c\}$, so $d(a, b) > d(a, c)$, *contradiction.*

Proposition 2.5.2

We refer here to Table 2.6 (page 95), "AGM interdefinability". Contraction, revision, and epistemic entrenchment are interdefinable by the following equations, i.e., if the defining side has the respective properties, so will the defined side. (See [AGM85].)

Speaking in terms of distance defined revision, $X \mid A$ is the set of those $a \in A$, which are closest to X, and $X \ominus A$ is the set of y which are either in X, or in $\boldsymbol{C}(A)$ and closest to X among those in $\boldsymbol{C}(A)$.

2.5.0.8 A remark on intuition

The idea of epistemic entrenchment is that ϕ is more entrenched than ψ (relative to K) iff $M(\neg\psi)$ is closer to $M(K)$ than $M(\neg\phi)$ is to $M(K)$. In shorthand, the more we can twiggle K without reaching $\neg\phi$, the more ϕ is entrenched. Truth is maximally entrenched - no twiggling whatever will reach falsity. The more ϕ is entrenched, the more we are certain about it. Seen this way, the properties of epistemic entrenchment relations are very natural (and trivial): As only the closest points of $M(\neg\phi)$ count (seen from $M(K)$), ϕ or ψ will be as entrenched as $\phi \wedge \psi$, and there is a logically strongest ϕ' which is as entrenched as ϕ - this is just the sphere around $M(K)$ with radius $d(M(K), M(\neg\phi))$.

Definition 2.5.2

$d : U \times U \to Z$ is called a pseudo-distance on U iff (d1) holds:

(d1) Z is totally ordered by a relation $<$.

If, in addition, Z has a $<-$smallest element 0, and (d2) holds, we say that d respects identity:

(d2) $d(a,b) = 0$ iff $a = b$.

If, in addition, (d3) holds, then d is called symmetric:

(d3) $d(a,b) = d(b,a)$.

(For any $a, b \in U$.)

Note that we can force the triangle inequality to hold trivially (if we can choose the values in the real numbers): It suffices to choose the values in the set $\{0\} \cup [0.5, 1]$, i.e., in the interval from 0.5 to 1, or as 0.

Definition 2.5.3

We define the collective and the individual variant of choosing the closest elements in the second operand by two operators, $|, \uparrow : \mathcal{P}(U) \times \mathcal{P}(U) \to \mathcal{P}(U)$:

Let d be a distance or pseudo-distance.

$X \mid Y := \{y \in Y : \exists x_y \in X . \forall x' \in X, \forall y' \in Y (d(x_y, y) \leq d(x', y'))\}$

(the collective variant , used in theory revision)

and

$X \uparrow Y := \{y \in Y : \exists x_y \in X.\forall y' \in Y(d(x_y, y) \leq d(x_y, y'))\}$

(the individual variant , used for counterfactual conditionals and theory update).

Thus, $A \mid_d B$ is the subset of B consisting of all $b \in B$ that are closest to A. Note that, if A or B is infinite, $A \mid_d B$ may be empty, even if A and B are not empty. A condition assuring nonemptiness will be imposed when necessary.

Definition 2.5.4

An operation $\mid: \mathcal{P}(U) \times \mathcal{P}(U) \to \mathcal{P}(U)$ is representable iff there is a pseudo-distance $d : U \times U \to Z$ such that

$A \mid B = A \mid_d B := \{b \in B : \exists a_b \in A \forall a' \in A \forall b' \in B(d(a_b, b) \leq d(a', b'))\}.$

The following is the central definition, it describes the way a revision $*_d$ is attached to a pseudo-distance d on the set of models.

Definition 2.5.5

$T *_d T' := Th(M(T) \mid_d M(T')).$

$*$ is called representable iff there is a pseudo-distance d on the set of models s.t. $T * T' = Th(M(T) \mid_d M(T'))$.

Fact 2.5.3

A distance based revision satisfies the AGM postulates provided:

(1) it respects identity, i.e., $d(a, a) < d(a, b)$ for all $a \neq b$,

(2) it satisfies a limit condition: minima exist,

(3) it is definability preserving.

(All conditions are necessary.)

Definition 2.5.6

We refer here to Table 2.7 (page 96), "Distance representation and revision". It defines the Loop Condition, and shows the correspondence between the semantic and the syntactic side.

The prerequisites are:

Let $U \neq \emptyset$, $\mathcal{Y} \subseteq \mathcal{P}(U)$ satisfy $(\cap)$, $(\cup)$, $\emptyset \notin \mathcal{Y}$.

Let $A, B, X_i \in \mathcal{Y}$, $\mid : \mathcal{Y} \times \mathcal{Y} \rightarrow \mathcal{P}(U)$.

Let $*$ be a revision function defined for arbitrary consistent theories on both sides. (This is thus a slight extension of the AGM framework, as AGM work with formulas only on the right of $*$.)

Proposition 2.5.4

Let $U \neq \emptyset$, $\mathcal{Y} \subseteq \mathcal{P}(U)$ be closed under finite $\cap$ and finite $\cup$, $\emptyset \notin \mathcal{Y}$.

(a) $\mid$ is representable by a symmetric pseudo-distance $d : U \times U \rightarrow Z$ iff $\mid$ satisfies $(\mid \mathit{Succ})$ and $(\mid \mathit{Loop})$ in Definition 2.5.6 (page 92).

(b) $\mid$ is representable by an identity respecting symmetric pseudo-distance $d : U \times U \rightarrow Z$ iff $\mid$ satisfies $(\mid \mathit{Succ})$, $(\mid \mathit{Con})$, and $(\mid \mathit{Loop})$ in Definition 2.5.6 (page 92).

See [LMS01] or [Sch04].

Proposition 2.5.5

Let $\mathcal{L}$ be a propositional language.

(a) A revision operation $*$ is representable by a symmetric consistency and definability preserving pseudo-distance iff $*$ satisfies $(*\mathit{Equiv})$, $(*\mathit{CCL})$, $(*\mathit{Succ})$, $(*\mathit{Loop})$.

(b) A revision operation $*$ is representable by a symmetric consistency and definability preserving, identity respecting pseudo-distance iff $*$ satisfies $(*\mathit{Equiv})$, $(*\mathit{CCL})$, $(*\mathit{Succ})$, $(*\mathit{Con})$, $(*\mathit{Loop})$.

See [LMS01] or [Sch04].

2.5.1 Theory revision for many-valued logics

To see what we have to do for distance based revision in the case of many-valued logics, we orient ourselves by the 2-valued case.

We considered sets X and Y, and were looking at pairs $\langle x, y \rangle$, $x \in X$, $y \in Y$ and choose those with minimal distance. So we compared $\langle x, y \rangle$ and $\langle x', y' \rangle$, $x, x' \in X$, $y, y' \in Y$, and $d(x, y)$ and $d(x', y')$. Thus, we now look pairs $\langle x, y \rangle$, $\langle x', y' \rangle$ again, and discard $\langle x, y \rangle$ if $d(x', y') < d(x, y)$, *and* "the value of $\langle x', y' \rangle$ is at least as good as the value of $\langle x, y \rangle$". To make the latter precise, we postulate, setting F for X, G for Y : $F(x) \leq F(x')$ and $G(y) \leq G(y')$. Thus, we consider

$\{\langle x, y \rangle$: $\neg \exists \langle x', y' \rangle.(d(x', y') < d(x, y)$ and ($(F(x) \leq F(x')$ and $G(y) \leq G(y'))$ or $(G(x) \leq G(x')$ and $F(y) \leq F(y'))))$ $\}$.

Table 2.5: AGM theory revision

AGM theory revision			
Contraction, $K - \phi$			
$(K-1)$	$K-\phi$ is deductively closed		
$(K-2)$	$K-\phi \subseteq K$	$(X \ominus 2)$	$X \subseteq X \ominus A$
$(K-3)$	$\phi \notin K \Rightarrow K-\phi = K$	$(X \ominus 3)$	$X \not\subseteq A \Rightarrow X \ominus A = X$
$(K-4)$	$\not\vdash \phi \Rightarrow \phi \notin K-\phi$	$(X \ominus 4)$	$A \neq U \Rightarrow X \ominus A \not\subseteq A$
$(K-5)$	$K \subseteq \overline{(K-\phi) \cup \{\phi\}}$	$(X \ominus 5)$	$(X \ominus A) \cap A \subseteq X$
$(K-6)$	$\vdash \phi \leftrightarrow \psi \Rightarrow K-\phi = K-\psi$		
$(K-7)$	$(K-\phi) \cap (K-\psi) \subseteq$ $K-(\phi \wedge \psi)$	$(X \ominus 7)$	$X \ominus (A \cap B) \subseteq$ $(X \ominus A) \cup (X \ominus B)$
$(K-8)$	$\phi \notin K-(\phi \wedge \psi) \Rightarrow$ $K-(\phi \wedge \psi) \subseteq K-\phi$	$(X \ominus 8)$	$X \ominus (A \cap B) \not\subseteq A \Rightarrow$ $X \ominus A \subseteq X \ominus (A \cap B)$
Revision, $K * \phi$			
$(K*1)$	$K*\phi$ is deductively closed	-	
$(K*2)$	$\phi \in K*\phi$	$(X \mid 2)$	$X \mid A \subseteq A$
$(K*3)$	$K*\phi \subseteq \overline{K \cup \{\phi\}}$	$(X \mid 3)$	$X \cap A \subseteq X \mid A$
$(K*4)$	$\neg\phi \notin K \Rightarrow$ $\overline{K \cup \{\phi\}} \subseteq K*\phi$	$(X \mid 4)$	$X \cap A \neq \emptyset \Rightarrow$ $X \mid A \subseteq X \cap A$
$(K*5)$	$K*\phi = K_\perp \Rightarrow \vdash \neg\phi$	$(X \mid 5)$	$X \mid A = \emptyset \Rightarrow A = \emptyset$
$(K*6)$	$\vdash \phi \leftrightarrow \psi \Rightarrow K*\phi = K*\psi$	-	
$(K*7)$	$K*(\phi \wedge \psi) \subseteq$ $\overline{(K*\phi) \cup \{\psi\}}$	$(X \mid 7)$	$(X \mid A) \cap B \subseteq$ $X \mid (A \cap B)$
$(K*8)$	$\neg\psi \notin K*\phi \Rightarrow$ $\overline{(K*\phi) \cup \{\psi\}} \subseteq K*(\phi \wedge \psi)$	$(X \mid 8)$	$(X \mid A) \cap B \neq \emptyset \Rightarrow$ $X \mid (A \cap B) \subseteq (X \mid A) \cap B$
Epistemic entrenchment			
$(EE1)$	$\leq_K$ is transitive	$(EE1)$	$\leq_X$ is transitive
$(EE2)$	$\phi \vdash \psi \Rightarrow \phi \leq_K \psi$	$(EE2)$	$A \subseteq B \Rightarrow A \leq_X B$
$(EE3)$	$\forall \phi, \psi$ $(\phi \leq_K \phi \wedge \psi$ or $\psi \leq_K \phi \wedge \psi)$	$(EE3)$	$\forall A, B$ $(A \leq_X A \cap B$ or $B \leq_X A \cap B)$
$(EE4)$	$K \neq K_\perp \Rightarrow$ $(\phi \notin K$ iff $\forall \psi . \phi \leq_K \psi)$	$(EE4)$	$X \neq \emptyset \Rightarrow$ $(X \not\subseteq A$ iff $\forall B . A \leq_X B)$
$(EE5)$	$\forall \psi . \psi \leq_K \phi \Rightarrow \vdash \phi$	$(EE5)$	$\forall B . B \leq_X A \Rightarrow A = U$

Table 2.6: AGM interdefinability

AGM interdefinability	
$K * \phi := \overline{(K - \neg\phi)} \cup \phi$	$X \mid A := (X \ominus \boldsymbol{C}(A)) \cap A$
$K - \phi := K \cap (K * \neg\phi)$	$X \ominus A := X \cup (X \mid \boldsymbol{C}(A))$
$K - \phi := \{\psi \in K : (\phi <_K \phi \vee \psi \text{ or } \vdash \phi)\}$	$X \ominus A := \begin{cases} X & iff \quad A = U, \\ \bigcap\{B : X \subseteq B \subseteq U, A <_X A \cup B\} & otherwise \end{cases}$
$\phi \leq_K \psi :\leftrightarrow \begin{cases} \vdash \phi \wedge \psi \\ or \\ \phi \notin K - (\phi \wedge \psi) \end{cases}$	$A \leq_X B :\leftrightarrow \begin{cases} A, B = U \\ or \\ X \ominus (A \cap B) \not\subseteq A \end{cases}$

Table 2.7: Distance representation and revision

Distance representation and revision		
		$(*Equiv)$ $\models T \leftrightarrow S, \models T' \leftrightarrow S', \Rightarrow T * T' = S * S'$,
		$(*CCL)$ $T * T'$ is a consistent, deductively closed theory,
	$(\mid Succ)$ $A \mid B \subseteq B$	$(*Succ)$ $T' \subseteq T * T'$,
	$(\mid Con)$ $A \cap B \neq \emptyset \Rightarrow A \mid B = A \cap B$	$(*Con)$ $Con(T \cup T') \Rightarrow T * T' = \overline{T \cup T'}$,
Intuitively, Using symmetry $d(X_0, X_1) \leq d(X_1, X_2)$, $d(X_1, X_2) \leq d(X_2, X_3)$, $d(X_2, X_3) \leq d(X_3, X_4)$ $\ldots$ $d(X_{k-1}, X_k) \leq d(X_0, X_k)$ $\Rightarrow$ $d(X_0, X_1) \leq d(X_0, X_k)$, i.e., transitivity, or absence of loops involving $<$	$(\mid Loop)$ $(X_1 \mid (X_0 \cup X_2)) \cap X_0 \neq \emptyset$, $(X_2 \mid (X_1 \cup X_3)) \cap X_1 \neq \emptyset$, $(X_3 \mid (X_2 \cup X_4)) \cap X_2 \neq \emptyset$, $\ldots$ $(X_k \mid (X_{k-1} \cup X_0)) \cap X_{k-1} \neq \emptyset$ $\Rightarrow$ $(X_0 \mid (X_k \cup X_1)) \cap X_1 \neq \emptyset$	$(*Loop)$ $Con(T_0, T_1 * (T_0 \vee T_2))$, $Con(T_1, T_2 * (T_1 \vee T_3))$, $Con(T_2, T_3 * (T_2 \vee T_4))$ $\ldots$ $Con(T_{k-1}, T_k * (T_{k-1} \vee T_0))$ $\Rightarrow$ $Con(T_1, T_0 * (T_k \vee T_1))$

Chapter 3

Towards a uniform picture of conditionals

3.1 Introduction

The word "conditional" contains "condition". So, a conditional is a structure of the form: "if condition c holds, then so will property p". Thus, conditionals seem to be at least always binary. But the condition may be hidden in additional structure (like a Kripke model relation, a preferential relation, etc.). Thus, we will also include unary structures.

But even if we restrict ourselves to binary (and why not ternary etc.?) conditionals, it seems that we can invent ad libitum new conditionals. We give two (arbitrary) examples, just to illustrate the possibilities, we do not pretend that they are very intuitive:

Example 3.1.1

(1) A new binary conditional: Suppose we have a distance d between models, and, in addition, a real valued function (e.g., of utility) f defined on the model set. Define now $m \models \phi > \psi$ iff in all $\phi-$worlds n, which are closest to m, and where f is locally constant, i.e., $f(n)$ does not change in a small $d-$neighbourhood around n, ψ holds.

(2) A new ternary conditional: Let again a distance d be defined. (ϕ, ψ, ϕ') holds iff in all worlds which are equidistant to the ϕ and ϕ' worlds (defined as in theory revision), ψ holds.

So it seems impossible to give an exhaustive enumeration of all possible conditionals.

We look at different possible classifications:

We may classify conditionals as to

(1) their arity,

(2) whether they are in the object language (like counterfactual conditionals), or in the meta-language (like usual preferential consequences), and if so, can we nest them, or do we run into trivialization results as for theory revision,

(3) whether they are based on classical logic, like usual modal logic, or perhaps some other logic,

(4) according to the properties of their semantic choice functions, like the properties of the $\mu-$functions of preferential structures,

(5) whether they work with *one* set of chosen models (as in the minimal variant of preferential structures), or with a $family$ of chosen sets (as in the limit version of preferential structures),

(6) whether they can be defined by binary relations, or perhaps some abstract size property, on the underlying model structure, like preferential or modal logic, or some higher relation, like distances between pairs of models, like for theory revision, update, or counterfactual conditionals,

(7) and if they are based on binary relations, according to the properties of those relations, like transitivity, smoothness, rankedness, etc.,

(8) whether we have a value function on the models, like utility, whether we have addition, and similar operations on these values,

(9) how the semantical structures are evaluated, e.g., both preferential and modal structures work with binary relations, but the relations are evaluated in totally different ways,

(10) whether we work with an unstructured language, or not, is there additional structure on the truth values, or not?,

Thus, also an exhaustive classification and listing of ordering principles seems quite hopeless to obtain.

In addition, for some conditionals, especially for modal logic, there is abundant literature, and we neither have exhaustive knowledge, nor do we think it important to summarize this literature here.

So, what will we do? Our purpose here is to begin to put some order into this multitude, or, perhaps even better, lay down some lines along which ordering is possible.

3.1.1 Overview of this chapter

3.1.1.1 Definition and classification

As argued in the introduction to this chapter, the best seems to be to say that a conditional is just any operator. Negation, conjunction, etc., are then included, but excluded from the discussion, as we know them well.

The classical connectives have a semantics in the boolean set operators, but there are other operators, like the $\mu-$functions of preferential logic which do not correspond to any such operator, and might even not preserve definability in the infinite case (see Definition 2.2.4 (page 42)). It seems more promising to order conditionals by the properties of their model choice functions, e.g., whether those functions are idempotent, etc., see Section 3.2.2 (page 101) .

Many conditionals can be based on binary relations, e.g. modal conditionals on accessibility relations, preferential consequence relations on preference relations, counterfactuals and theory revision on distance relations, etc. Thus, it is promising to look at those relations, and their properties to bring more order into the vast field of conditionals. D.Gabbay introduced reactive structures (see, e.g., [Gab04]), and added supplementary expressivity to structures based on binary relations, see [GS08b] and [GS08f]. In particular, it was shown there that we can have cumulativity without the basic properties of preferential structures (e.g., OR). This is discussed in Section 3.2.4 (page 102).

3.1.1.2 Additional structure on language and truth values

Normally, the language elements (propositional variables) are not structured. This is somewhat surprising, as, quite often, one variable will be more important than another. Size or weight might often be more important than colour for physical objects, etc. It is probably the mathematical tradition which was followed too closely. One of the authors gave a semantics to theory revision using a measure on language elements in [Sch91-1] and [Sch91-3], but, as far as we know, the subject was not treated in a larger context so far. The present book often works with independence of language elements, see in particular Chapter 4 (page 125) and Chapter 5 (page 165), and Hamming type relations and distances between models, where it need not be the case that all variables have the same weight. Thus, it is obvious to discuss this subject in the present text. It can also be fruitful to discuss sizes of subsets of the set of variables, so we may, e.g., neglect differences to classical logic if they concern only a "small" set of propositional variables.

On the other hand, classical truth values have a natural order, $FALSE < TRUE$, and we will sometimes work with more than 2 truth values, see in particular Chapter 4 (page 125) , but also Section 5.3.6 (page 203). So there is a natural question: do we also have a total order, or a boolean order, or another order on those sets of truth values? Or: Is there a distance between truth values, so that a change from value a to value b is smaller than a change from a to c?

There is a natural correspondence between semantical structures and truth values, which is best seen by an example: Take finite (intuitionistic) Goedel logics, see Section 4.4.3 (page 148), say, for simplicity with two worlds. Now, ϕ may hold nowhere, everywhere, or only in the second world (called "there", in contrast to "here", the first world). Thus, we can express the same situation by three truth values: 0 for nowhere, 1 for only "there", 2 for everywhere.

In Section 3.3.6 (page 107), we will make some short remarks on "softening" concepts, like neglecting "small" fragments of a language, etc. This way, we can define, e.g., "soft" interpolation, where we need a small set of variables which are not in both formulas.

Inheritance systems, (see, e.g., [TH89], [THT86], [THT87], [TTH91], [Tou86], also [Sch93] and [Sch97]), present many aspects of independence, (see Section 3.3.7 (page 107)). Thus, if two nodes are not connected by valid paths, they may have very different languages, as language elements have to be inherited, otherwise, they are undefined. In addition, a may inherit from b property c, but not property d, as we have a contradiction to d (or, even $\neg d$) via a different node b'. Theses are among the aspects which make them natural, but also quite different from traditional logics.

3.1.1.3 Representation for general revision, update, and counterfactuals

Revision (see [AGM85], and the discussion in Section 2.5 (page 90)), update (see [KM90]), and counterfactuals (see [Lew73] and [Sta68]) are special forms of conditionals, which received much interest in the artificial intelligence community. Explicitly or implicitly (see [LMS95], [LMS01]), they are based on a distance based semantics, working with "closest worlds". In the case of revision, we look at those worlds which are closest to the present *set* of worlds, in update and counterfactual, we look from each present world *individually* to the closest worlds, and then take the union. Obviously, the formal properties may be very different in the two cases.

There are two obvious generalizations possible, and sometimes necessary. First, “closest” worlds need not exist, there may be infinite descending chains of distances without minimal elements. Second, a distance or ranked order may force too many comparisons, when two distances or elements may just simply not be comparable. We address representation problems for these generalizations:

(1) We first generalize the notion of distance for revision semantics in Section 3.4.3 (page 112). We mostly consider symmetrical distances, so $d(a, b) = d(b, a)$, and we work with equivalence classes $[a, b]$. Unfortunately, one of the main tools in [LMS01], a loop condition, does not work any more, it is too close to rankedness.

   We will have to work more in the spirit of general and smooth preferential structures to obtain representation. Unfortunately, revision does not allow many observations (see [LMS01], and, in particular, the impossibility results for revision (“Hamster Wheels”) discussed in [Sch04]), so all we have (see Section 3.4.3.3 (page 114)) are results which use more conditions than what can be observed from revision observations. This problem is one of principles: we showed in [GS08a], see also [GS08f], that cumulativity suffices only to guarantee smoothness of the structure if the domain is closed under finite unions. But the union of two products need not be a product any more.

   To solve the problem, we use a technique employed in [Sch96-1], using “witnesses” to testify for the conditions.

(2) We then discuss the limit version (when there are no minimal distances) for theory revision.

(3) In Section 3.4.4 (page 117), we turn to generalized update and counterfactuals. To solve this problem, we use a technique invented in [MS90], and adapt it to our situation. The basic idea is very simple: we begin (simplified) with some world x, and arrange the other worlds around x, as x sees them, by their relative distances. Suppose we consider now one those worlds, say y. Now we arrange the worlds around y, as y sees them. If we make all the new distances smaller than the old ones, we “cannot look back”, etc. We continue this construction unboundedly (but finitely) often. If we are a little careful, everyone will only see what he is supposed to see. In a picture, we construct galaxies around a center, then planets around suns, moon around planets, etc.

   The resulting construction is an $\mathcal{A}-$ranked structure, as discussed in [GS08d], see also [GS08f].

(4) In Section 3.4.5 (page 122), we discuss the corresponding syntactic conditions, using again ideas from [Sch96-1].

3.2 An abstract view on conditionals

3.2.1 A general definition as arbitrary operator

Definition 3.2.1

(1) The 2-valued case:

Let M be the set of models for a given language $\mathcal{L}$.

(1.1) Single sets:

A $n-$ary semantical conditional C is a $n-$ary function

$C : \mathcal{P}(M) \times \ldots \times \mathcal{P}(M) \to \mathcal{P}(M)$.

(1.2) Systems of sets:

A $n-$ary semantical sys-conditional C is a $n-$ary function

$C : \mathcal{P}(M) \times \ldots \times \mathcal{P}(M) \to \mathcal{P}(\mathcal{P}(M))$.

(2) The many-valued case:

Let V be the set of truth values, L the set of propositional variables of $\mathcal{L}$, let M be the set of functions $m : L \to V$ (such m are the generalizations of a 2-valued model). Let $\mathcal{M}$ be the set of functions $F : M \to V$ (such F are the generalization of a model set).

(2.1) Single "sets":

A $n-$ary many-valued semantical conditional C is a $n-$ary function

$C : \mathcal{M} \times \ldots \times \mathcal{M} \to \mathcal{M}$.

(2.2) Systems of "sets":

A $n-$ary many-valued semantical sys-conditional C is a $n-$ary function

$C : \mathcal{M} \times \ldots \times \mathcal{M} \to \mathcal{P}(\mathcal{M})$.

Note that the definition is not yet fully general, as we might have sets of pairs of elements as a result, e.g., the pairs with minimal distance from $X \times Y$ - but this might cause more confusion than clarity.

When the context is clear, we will just speak of conditionals, without any further precision.

Example 3.2.1

(1) Negation (complement) $C : M(\phi) \mapsto M(\neg\phi)$ is an unary 2-valued conditional.

(2) AND (intersection) $C : \langle M(\phi), M(\psi)\rangle \mapsto M(\phi \wedge \psi)$ is a binary 2-valued conditional.

(3) Given a preferential relation on M, defining the sets of minimal elements $\mu(X)$ of X, μ is an unary 2-valued conditional.

(4) Given a preferential relation on M, defining the systems of $MISE$'s, $MISE : X \mapsto \{Y \subseteq X : Y$ is MISE in $X\}$ is an unary 2-valued sys-conditional.

(5) Often, the same underlying structure can define a simple or a system conditional: A preferential relation generates the set of minimal elements, and the system of MISE.

(6) The same underlying structure may also be used totally differently, e.g., a binary relation may be used to find the minimal or the accessible elements.

3.2.2 Properties of choice functions

We may classify conditionals by the abstract properties of their choice functions:

(1) The $\mu-$functions of general preferential structures obey additive size laws, as described in Table 5.1 (page 189) and Table 5.2 (page 190).

(2) Cumulative or smooth structures obey additional laws.

(3) Rational structures obey the rule that $\mu(A \cup B) = \mu(A) \cup \mu(B)$ or $\mu(A \cup B) = \mu(A)$ or $\mu(A \cup B) = \mu(B)$.

(4) For the usual preferential structures, we have $\mu(\mu(X)) = \mu(X)$.

(5) Modular structures show multiplicative rules, as described in Table 5.3 (page 191).

(6) Update and counterfactual conditionals are additive on the left: $\mu(A \cup B, X) = \mu(A, X) \cup \mu(B, X)$.

3.2.3 Evaluation of systems of sets

Note that, even if the set or system of sets (in the 2-valued case) is fully described, it is not yet fully defined what we do with the information, not even in the finite (and thus definability preserving) case:

Given the system of sets, we can

(1) determine what holds in all elements of the set, or finally in the "good" sets, see Section 2.3.2 (page 61),

(2) describe exactly those sets, e.g. in:

- Boutelier's modal logic approach to preferential structures, see [Bou90a],
- the "good" sets of deontic logic, see [GS08f] and Chapter 6 (page 213).

Such systems are not necessarily well described as neighbourhood systems, and are usually not yet well examined.

3.2.4 Conditionals based on binary relations

Often, C is determined by a binary relation, and we can try to classify the conditionals by the properties of the relation and the way the relation is used.

Example 3.2.2

(1) Kripke structures for modal, intuitionistic, and other logics,

(2) preferential structures,

(3) distance based theory revision (the relation is on the Cartesian product),

(4) the Stalnaker-Lewis semantics for counterfactual conditionals, and distance based update (again, the relation is on the Cartesian product, but used differently).

But we can also see the following structures as given by a binary relation:

(5) Defaults: A default $A \to A \wedge B$ (we take only simple examples) is pure defense, $A \wedge B-$models are "better" than $A \wedge \neg B-$models, we do *not* say that $A \wedge B-$models attack $A \wedge \neg B-$models. (This view can have repercussions when chaining defaults.) The "best" situations are those which satisfy a maximum of default rules. See Chapter 6 (page 213).

(6) Obligations: An obligation can be seen the same way as a default. See again Chapter 6 (page 213).

(7) Contrary to duty obligations can also be seen as such rules. But here, the best situations are those which satisfy the primary obligation, etc. - this is *not* an order by specificity.

We can then classify conditionals as to the properties of these relations, like transitivity, smoothness, etc., and their use.

3.2.4.1 Short discussion of above examples

Obviously, the relation of accessibility in Kripke models is one of pure defense: if xRy, then x "supports" or "defends" y. This results in the trivial property

$X \subseteq Y \Rightarrow C(X) \subseteq C(Y)$

The situation in preferential structures is more complicated: If $x \in X$, then x "defends" itself. There is no defense of another element, but there is attack, if $x \prec y$, then x attacks y. The situation is, as a matter of fact, still more subtle. Preferential structures often work with copies. Each copy $\langle x, i \rangle$ is a defense of x. This becomes obvious, when we think that, to destroy minimality of x, we have to destroy *all* copies of x.

As there is no attack on attacks, attacks have a property of monotony, what is attacked in X, is also attacked in any $Y \supseteq X$. This results in:

$X \subseteq Y \Rightarrow C(Y) \cap X \subseteq C(X)$.

Smoothness means, roughly, that, if there is an attack, there is an attack from a valid element (or copy).

Higher preferential structures may attack attacks, which is then a defense, see Section 2.4.2 (page 72).

In Example 3.2.2 (page 102), (3) and (4), we have a ranking of distances. Thus, any $\langle x, y \rangle$ is a defense of y (when we consider the second coordinate), and an attack on bigger pairs $\langle x', y' \rangle$.

Consequently, we may also classify conditionals on the role of support and attack:

- there are no conflicts, e.g.

  we have pure defense, as in Kripke models,

  we have pure attack, as in classical preferential structures

- conflicts are resolved or not e.g. by sup of the absolute values of the strength of support and attack, when this exists, by addition, if there is + on Y, etc.:
  - in reactivite diagrams, we may consider the biggest absolute values for $f(x, y')$ for fixed y', and $x \in A$: basic arrows are attack of force -1, attack on attack has force +2, etc.
  - in inheritance diagrams, we use the valid paths for strength comparison (see [GS08e], [GS08f]),

– so far, the use of support and attack does not seem to use ressources, as is the case in Girard's Linear Logic, it remains an open research problem to investigate utility and properties of such approaches.

3.3 Conditionals and additional structure on language and truth values

3.3.1 Introduction

This section gives only a very rough outline of the possible approaches.

We can treat the set of language elements (propositional variables in our context), or of truth values, like any other set. Thus, we can introduce relations, operations, etc. on those sets. In addition, we can compose sublanguages into bigger languages, small truth value sets in bigger one, or, conversely, go from bigger to smaller sets, etc.

This section is mostly intended to open the discussion. What is reasonable to do, will be shown when necessity arises from applications.

3.3.2 Operations on language and truth values

We can consider structures and operations on

- language elements (propositional variables) and truth values within one language
- languages and truth value sets for several, different languages
- definable model sets.

We will take now a short look at the different cases.

3.3.3 Operations on language elements and truth values within one language

Operations on language elements

(1) A relation of importance between language elements:

(1.1) If p is more important than q, then we might give "biased" weight e.g. in theory revision, see [Sch95-2], [Sch91-1], and [Sch91-3].

(1.2) This might lead to constructing a modular order of models, for non-monotonic logic and theory revision, e.g., by a lexicographic order, where the more important language elements have precedence. Suppose for instance that a is considered more important than b, but that we prefer a and b over their negation. This might lead to the following order: $\neg a \neg b < \neg ab < a \neg b < ab$.

(2) A distance relation between language elements

(2.1) See Section 3.3.7 (page 107), for the structuring of the language in inheritance diagrams via a distance between language elements.

(3) Size of sets of language elements:

(3.1) A notion of "small" sets of language elements allows to introduce "soft" concepts (see also below, Section 3.3.6 (page 107)):

(3.1.1) Soft cumulativity: for each model, there is a model which differs only on a small set of variables, and which is minimized by a minimal model.

(3.1.2) Soft modularity and independence: being modular, independent with some exceptions.

(3.2) We can define model distance by the size of elements by which they differ, they are "almost" identical if this set is small.

(3.3) If J is a small subsets of L (the set of propositional variables), then we have (in classical logic) $2^J = 2^{small}$ models, so we can work with rules about abstract exponentiation.

(4) Accessibility of language elements:

(4.1) Consider inheritance systems: in the Nixon diamond, it is undecided if Nixon is a pacifist, this can now be distinguished from the case where there no path whatsoever from A to B : language element B (pazifist) is not reachable from language element A (Nixon).

(4.2) Consider natural language and argumentation: constructing an argument or a scenario, adding language elements when necessary, and thus keeping representation simple and efficient, introducing only what is needed.

(5) Combination of language elements:

We may combine several language elements to one "super-element" and thus achieve abstraction, like all properties for colour are grouped under "colour". Conversely, we may differentiate one element into sub-elements, see also "bubble structures", Section 2.3.1.1 (page 58).

(6) Context dependency:

All above considerations may be context dependent in more complicated situations, as in inheritance or argumentation.

Operations on truth values

(1) The classical truth value set is a Boolean algebra. Finite Goedel logics (see Section 4.4.3 (page 148)) can be seen as having a linearly ordered truth value set, see below, (5).

(2) We can consider more general Boolean algebras, or just partial orders. In the case of argumentation, we may just consider the powerset of the arguments as the set of truth values - together with the usual operations.

(3) We may combine several truth values to one "super-value" and thus achieve abstraction, e.g., if the truth values are arguments. For instance, all arguments from one source may be grouped together. Conversely, we may differentiate one truth value into sub-values.

(4) We may define a notion of size on the set of truth values.

(5) Note that we have an equivalence between structure on the model set and structure on the truth value set:

Consider a Kripke structure. We have an equivalent re-formulation when we look at the same structure on the truth value set. ϕ is true in world m, iff the truth value m for ϕ is true. Intuitionistic logic is then a condition about truth values, not about Kripke structures. (See also Section 4.4.3 (page 148).)

These connections could be formally expressed by isomorphy propositions.

It might be reasonable to work sometimes with mixed structures. If a substructure is repeated everywhere, we can put it into the truth values, so every point of the simplified structure is differentiated by the truth value structure. This is, abstractly, similar to the "bubble structures" introduced in Section 2.3.1.1 (page 58), with identical bubbles.

3.3.4 Operations on several languages

Operations on language elements

(1) We may compose a bigger language from sub-languages, or, conversely, split bigger languages into sub-languages.

(2) The projection (respectively the inf and sup operators of Section 4.3 (page 136)) are new operators on model sets, but also on the language, going from a formula in a richer language to one in a poorer language.

(3) In argumentation and inheritance, we can introduce new language elements: "just think of ....", or by establishing a valid path upward from a given node in inheritance networks.

(4) Note that the structures on a language L need not be coherent with structures on $L' \subseteq L$, see Chapter 5 (page 165).

Operations on truth values

In defeasible inheritance, we can consider accessible nodes as truth values (with valid paths for comparison). Thus, we *construct* the truth value structure while evaluating the net above a certain point. Likewise, we may consider arguments for a formula as truth values, which are then constructed during the process.

3.3.5 Operations on definable model sets

We have to distinguish:

(1) Internal operators of the language, like the classical operators AND, they have their natural interpretation in boolean set operators.

(2) External operators like the choice operator of preferential structures μ.

(3) Algebraic operators, like projection (respectively the inf and sup operators of Section 4.3 (page 136)), guaranteeing syntactic interpolation.

3.3.6 Softening concepts

There are several ways we can "soften" a concept.

(1) If we have a many-valued logic, we can go down from maximal truth.

(2) We can sometimes replace rules by weaker versions, e.g., $\phi \mid\!\sim \psi$ by $\phi \mid\!\not\sim \neg\psi$.

(3) We can neglect small fragments of the language, e.g.,

- instead of having full interpolation like $\phi \mid\!\sim \alpha \mid\!\sim \psi$ with α having only symbols common to both ϕ and ψ, α is now allowed to have "some, but not too many" symbols which are not common to both ϕ and ψ,
- strictly speaking, $\alpha \mid\!\not\sim \beta$, but if we consider a big subset of the language, and α' is the fragment of α in that language, likewise for β', then $\alpha' \mid\!\sim \beta'$.

(4) "Soft" independence, where Σ is "almost" $\Sigma' \times \Sigma''$, in several interpretations:

- neglecting a small set of variables, we can write Σ as a product
- neglecting a small set of sequences, we can write Σ as a product, e.g., when $\Sigma = (\Sigma' \times \Sigma'') \cup \{\sigma\}$.
- If we have a distance between language elements, then we can express that, the more language elements are distant from each other, the more they are independent.
- We may have independence for small changes, but not for big ones.

  A small change may be to go from "big" to "medium" size, a big change to go from "big" to "small", etc.

(5) Generally, we can "soften" an object by considering another object, which differs only slightly from the original one (how ever this is measured). A concept can then be softened by softening the properties, or the objects involved.

It seems premature to go into details here, which should be motivated by concrete problems.

3.3.7 Aspects of modularity and independence in defeasible inheritance

Inheritance structures have many aspects, they are discussed in detail in [GS08e], see also [GS08f].

We discuss here the following, illustrated in Diagram 3.3.1 (page 108), and Diagram 3.3.2 (page 110): the left hand side shows the diagram, the right hand side the strength of information. E.g., in part (2), the strongest information available at A (except A itself), is $B \wedge C$. But there might be exceptions, and the most exceptional situation is $\neg B \wedge \neg C$.

(1) We can see the language as being constructed dynamically. In (1) $A \to C$, $B \to C$, A knows nothing about B and vice versa, C knows nothing about A or B. Giving a truth value "undecided" would not be correct. We are not decided about Nixon's pacifism in the Nixon Diagram, but we have (contradictory) information about it. Here we have *no* information.

Consider also Diagram 3.3.2 (page 110). As there is no monotonous path whatever between e and d, the question whether e's are d's or not, or vice versa, does not even arise. For the same reason, there is no question whether b's are c's, or not. In upward chaining formalisms, as there is no valid positive path from a to d, there is no question either whether a's are f's or not.

(2) We can see inheritance diagrams as Kripke structures, where the accessibility relation is constructed non-monotonically, it is not static.

(3) We can see inheritance diagrams as constructing a structure of truth values dynamically, along with the relation of comparison between truth values. This truth value structure is not absolute, but depends on the node from which we start. (A strongest element is always given, the information given directly at each node. The relation between the others is decided by specificity via valid paths.)

(4) Information is given independently, A might have many normal properties of B, but not all of them.

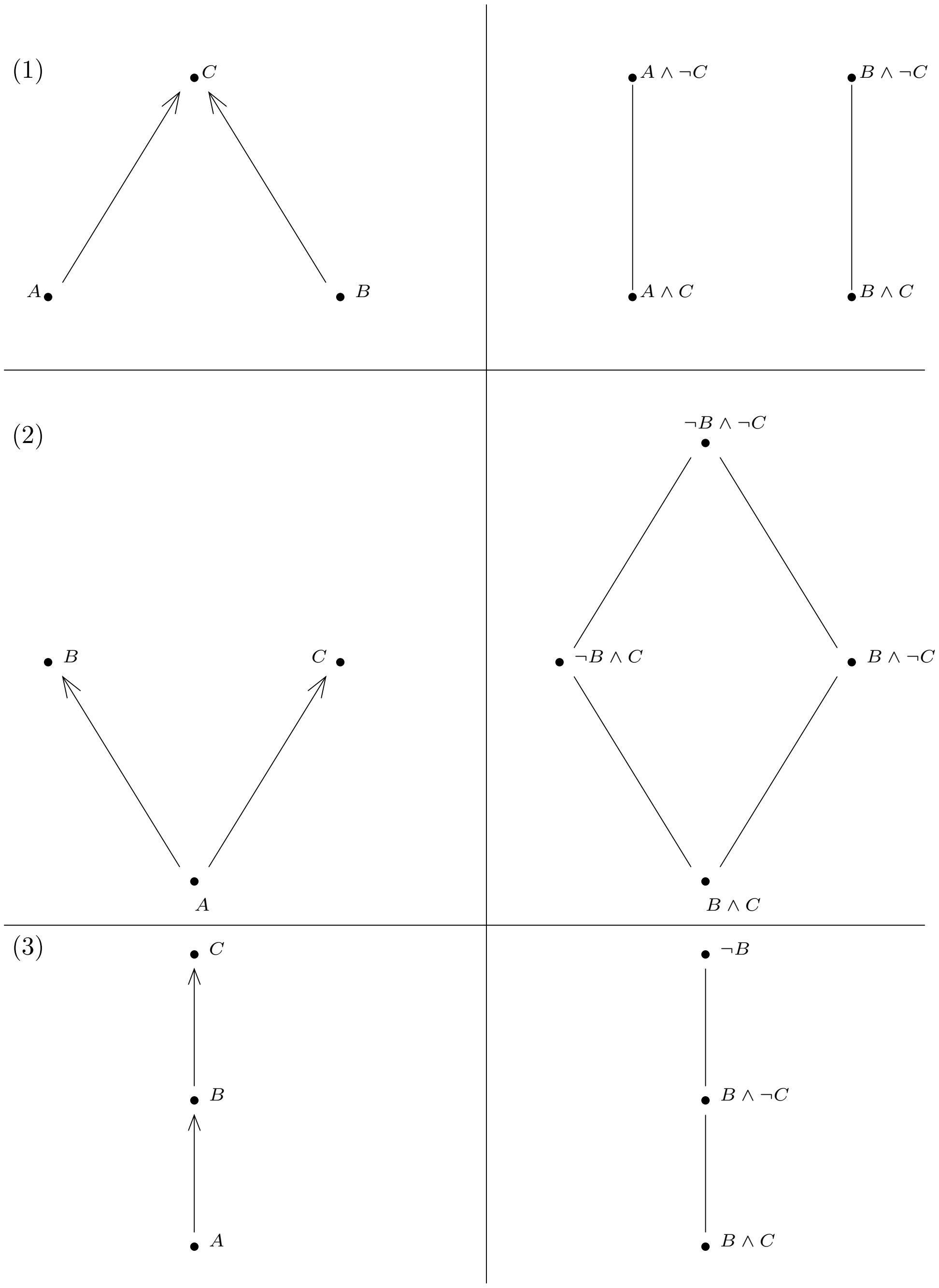
In (2) of Diagram 3.3.1 (page 108) $A \to B$, $A \to C$, and the information is independent. The resulting order is, seen from $A: B \wedge C < B \wedge \neg C < \neg B \wedge \neg C$, $B \wedge C < \neg B \wedge C < \neg B \wedge \neg C$.

Consider now (3): $A \to B \to C$. The resulting order is $B \wedge C < B \wedge \neg C < \neg B$. $\neg B$ is not differentiated any more, we have no information about it.

Consider again Diagram 3.3.2 (page 110). In our diagram, a's are b's, but not ideal b's, as they are not d's, the more specific information from c wins. But they are e's, as ideal b's are. So they are not perfectly ideal b's, but as ideal b's as possible. Thus, we have graded ideality, which does not exist in preferential and similar structures. In those structures, if an element is an ideal element, it has all properties of such, if one such property is lacking, it is not ideal, and we can't say anything any more beyond classical logic. Here, however, we sacrifice as little normality as possible, it is thus a minimal change formalism.

(5) Inheritance diagrams also give a distance between language elements, and thus a structure on the language: In the simplest approach, if the (valid) path from A to B is long, then the language elements A and B are distant. A better approach is via specificity, as usual in inheritance diagrams.

Diagram 3.3.1


(1)
C
A
B
$A \wedge \neg C$
$B \wedge \neg C$
$A \wedge C$
$B \wedge C$
(2)
$\neg B \wedge \neg C$
B
C
$\neg B \wedge C$
$B \wedge \neg C$
A
$B \wedge C$
(3)
C
$\neg B$
B
$B \wedge \neg C$
A
$B \wedge C$

$$\begin{tikzcd}
 & f & \\
e & d \arrow[u] & \\
b \arrow[u] \arrow[ur] & & c \arrow[ll] \arrow[ul, "/" marking] \\
 & a \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Diagram 3.3.2 Information transfer

3.4 Representation for general revision, update, and counterfactuals

3.4.1 Importance of theory revision for general structures, reactivity, and its solution

Theory revision is about minimal change. When we revise a theory with a formula, we obtain a new theory (in the AGM approach). When we want to revise other objects, like one logic with another logic, we may face the following problems:

(1) We have to choose an adequate level of representation: For instance, if we revise one logic with another logic, do we represent the logics

  (1.1) syntactically, as a relation between formulas, or

  (1.2) semantically, e.g. by preferential structures, or

  (1.3) by their abstract semantics, e.g., the sets of big subsets in preferential logics?

  When we chose a syntactic representation, we have to make the approach robust under syntactic reformulations (as in AGM revision). The same holds when we choose representation by preferential structures, as the same logic can be represented by many different structures, (for instance, by playing with the number of copies). We can, of course, choose a canonical representation, but this risks to be artificial. Probably the best approach here is to take the abstract semantics with big subsets.

(2) Traditional revision has a limit condition, by which (semantically speaking) closest models will always exist. It is not at all clear that such a condition will always be satisfied. For instance, there will often not be a closest real-valued continuous function with certain properties, but only closer and closer ones. Thus, we will have to take a limit approach, instead of the traditional minimal approach.

(3) It will often be too strong to require a ranked order on the elements, as AGM revision presupposes. One object may be closer in one aspect, another object in another object, but this closeness is not comparable. So, in general, we will only have a partial order.

(4) The closest object may not have the desired properties. E.g., we take transitive relation, modify it minimally by adding one pair to the relation, then the relation will not always be transitive any more. Thus, structural properties might not be preserved.

Due to the importance of theory revision, we will now give new representation results, which address some of these problems.

3.4.2 Introduction

We work here on representation for not necessarily ranked orders for distance based revision, and for update/counterfactual conditionals.

We do first the general, and then the smooth case for revision, then the same for update/counterfactual conditionals. We also examine the limit versions of revision and update/counterfactual conditionals.

Recall from Table 2.4 (page 59) the semantic representation results for general preferential structures. We will use them now. The complication of our situation lies in the fact that we cannot directly observe the "best" elements (which are now pairs), but only their projections. In the case of ranked structures, representation as shown in [LMS01] was easier, as, in this case, any minimal element was comparable with any non-minimal element, but this is not the case any more.

3.4.3 Semantic representation for generalized distance based theory revision

3.4.3.1 Description of the problem

We characterized distance based revision in [LMS01]. Our aim is to generalize the result to more general notions of distance, where the abstract distance is just a partial order, and also to cases where the limit condition, i.e., there are always closest elements, may be violated.

The central characterizing condition in [LMS01] was an elegant loop condition, due to M.Magidor. Unfortunately, this condition seems to be closely related to the rankedness condition, which we do not have any more. (More precisely, any minimal element was comparable to any non-minimal element, now a minimal and a non-minimal element need not be comparable. Even in the smooth case, a non-minimal element has to be comparable only to *one* minimal element, but not to all minimal elements.) We do not see how to find a similar nice condition in our more general situation. (The loop condition would have to be modified anyway, but this is not the main problem: $(X \mid (Y \cup Y')) \cap Y \neq \emptyset$ has to be changed to: $(X \mid (Y \cup Y')) \cap Y' = \emptyset$ and $(X \mid (Y \cup Y')) \subseteq Y$.)

We treated in the past a problem similar to the present one in [Sch96-1]. There, the existence of (at least) one minimal comparable element was captured by the idea of "witnesses", described by a formula $\phi(\langle a, b\rangle, A \times B)$, which expresses that $\langle a, b\rangle$ is a valid candidate not only for $A \times B$ but also for all $A' \times B'$, $A' \subseteq A$, $B' \subseteq B$. This gives the main condition for preferential representation. Another paper treating somehow similar problems was [BLS99], where we worked with "patches".

The idea pursued here is the same as in [Sch96-1]. We only isolate the different layers of the problem better, so our approach is more flexible for various conditions imposed on the abstract distance. This allows us to treat many variants in a very general way, mostly just citing the general representation results, and putting things together only at the end.

(1) Thus, we first give a general machinery for treating symmetrical distances by a translation to equivalence classes. See Section 3.4.3.2 (page 113).

(2) We then consider representation for the equivalence classes, with various additional conditions like smoothness/semantical cumulativity, translating them back to conditions about the original pairs $\langle a, b\rangle$. See Section 3.4.3.3 (page 114).

(3) In the next step, we see how much we can do for the original problem, where we do not have arbitrary $A \subseteq U \times U$ (U is the universe we work in), but only sets of the type $A \times B$, $A, B \subseteq U$. In particular, the union of $A \times B$ and $A' \times B'$ need not be again such a product. Moreoever, we can observe only the projection of $\mu(A \times B)$ onto the second coordinate:

$\{b \in B : \exists\langle a,b\rangle \in \mu(A \times B)\}$, see Section 3.4.3.4 (page 116). More precisely, we look for $\mu(Z)_X := \{x \in X : \exists y \in Y.\langle x,y\rangle \in \mu(Z)\}$ and $\mu(Z)_Y := \{y \in Y : \exists x \in X.\langle x,y\rangle \in \mu(Z)\}$. Of course, $x \in \mu(Z)_X$, $y \in \mu(Z)_Y$ does not imply $\langle x,y\rangle \in \mu(Z)$. We only know that for all $x \in \mu(Z)_X$ there is at least one $y \in \mu(Z)_Y$ such that $\langle x,y\rangle \in \mu(Z)$, and conversely. Thus, in the original problem, we have less input, and also less observation of the output. The implications of scarce information for representation was, for example, illustrated in [Sch04], by "Hamster Wheels", where we showed that this might make finite characterizations impossible.

(4) In a final step, we will go from the semantic side to the descriptive, logical side. See Section 3.4.5 (page 122).

3.4.3.2 Symmetrical distances and translation to equivalence classes

We work in some universe U. We want to express that the distance from a to b is the same as the distance form b to a. So, when pairs stand for distance, $\langle a,b\rangle$ and $\langle b,a\rangle$ will be equivalent, and we consider equivalence classes $[a,b]$ instead of pairs $\langle a,b\rangle$. The $[a,b]$'s will be the abstract distances, not the $\langle a,b\rangle$'s.

We define

Definition 3.4.1

Let $A, A' \subseteq U \times U$.

(1) $A \sim A'$ iff $\forall\langle a,b\rangle \in A(\langle a,b\rangle \in A'$ or $\langle b,a\rangle \in A')$ and $\forall\langle a,b\rangle \in A'(\langle a,b\rangle \in A$ or $\langle b,a\rangle \in A)$.

(2) $[A] := \{[a,b] : \langle a,b\rangle \in A$ or $\langle b,a\rangle \in A\}$

Note that we might lose elements when going from A to $[A]$, precisely when both $\langle a,b\rangle \in A$ and $\langle b,a\rangle \in A$, then both are in the class $[a,b]$.

We have

Fact 3.4.1

$A \sim A'$ iff $[A] = [A']$.

Proof

Trivial. □

We suppose a choice function μ to be given, chosing pairs of minimal distance, more precisely $\mu(A) := \{\langle a,b\rangle \in A : \forall\langle a',b'\rangle \in A.d(a,b) \leq d(a',b')\}$. We try to represent μ by a suitable ordering relation $\prec$ on pairs. As the distance relation is supposed to be symmetric, we will work with equivalence classes, considering $\mu'([A])$, instead of $\mu(A)$, and representing μ' instead of μ. We have to define μ' from μ, and make sure that the definition is independent of the choice of the particular element we work with. For this to work, we consider the following axiom $(\mu S1)$ about symmetry of distance. We also add immediately an axiom $(\mu S2)$ which says that, if both $\langle a,b\rangle$ and $\langle b,a\rangle$ are present, then both or none are minimal.

Definition 3.4.2

We define two conditions about symmetrical μ :

$(\mu S1)$ Let $A, A' \subseteq U \times U$.

If $A \sim A'$, then $\{[a,b] : \langle a,b\rangle \in \mu(A)$ or $\langle b,a\rangle \in \mu(A)\} = \{[a,b] : \langle a,b\rangle \in \mu(A')$ or $\langle b,a\rangle \in \mu(A')\}$.

$(\mu S2)$ Let $A \subseteq U \times U$, and both $\langle a,b\rangle, \langle b,a\rangle \in A$.

Then $\langle a,b\rangle \in \mu(A)$ iff $\langle b,a\rangle \in \mu(A)$.

Axiom $(\mu S1)$ says that it is unimportant for μ in which form $[a,b]$ is present, as $\langle a,b\rangle$, $\langle b,a\rangle$, or as both.

The main definition is now:

Definition 3.4.3

$\mu'([A]) := \{[a,b] \in [A] : \langle a,b\rangle \in \mu(A)$ or $\langle b,a\rangle \in \mu(A)\}$.

We have to show that μ' is well-defined, i.e. the following Fact holds.

Fact 3.4.2

If $(\mu S1)$ holds, and $[A] = [B]$, then $\mu'([A]) = \mu'([B])$.

Proof

If $[A] = [B]$, then by Fact 3.4.1 (page 113) $A \sim B$, so by $(\mu S1)$ $\{[a,b] : \langle a,b\rangle \in \mu(A)$ or $\langle b,a\rangle \in \mu(A)\}$ $= \{[a,b] : \langle a,b\rangle \in \mu(B)$ or $\langle b,a\rangle \in \mu(B)\}$, and by Definition 3.4.3 (page 114) $\mu'([A]) = \mu'([B])$. $\Box$

Suppose now we have a representation result for μ', i.e. some relation $\prec$ such that (simplified, without copies):

$\mu'([A]) = \{[a,b] \in [A] : \neg\exists[a',b'] \in [A].[a',b'] \prec [a,b]\}$.

Recall that $[a,b]$ is the (abstract) symmetrical distance between a and b. Then the relation $\prec$ describes those pairs $\langle a,b\rangle \in A$ which have minimal distance, more precisely, all $\langle a,b\rangle$ or $\langle b,a\rangle$ such that $[a,b] \in [A]$. So our representation problem is solved.

We may want to impose additional conditions. They may come from μ', where, e.g., we may want semantic cumulativity, they have then to be translated back to conditions about μ. They may come from requirements about the distance, e.g., that all distances from x to x are minimal, they have to be translated to conditions for μ'. We may also have conditions about the domain, e.g., we do not consider all $A \subseteq U \times U$, but only some, so we work on some $\mathcal{Y} \subseteq \mathcal{P}(U \times U)$, which may have some conditions like closure under finite unions - or, conditions on the sets of equivalence classes to be considered. Such additional conditions will be considered below.

3.4.3.3 General representation results and their translation

The general and smooth case

We continue to work in U, but with the equivalence classes. Let $[U] := \{[a,b] : a,b \in U\}$, and $\mathcal{Y} \subseteq [U]$. For the smooth case, let $\mathcal{Y}$ be closed under finite unions.

Consider $\mu' : \mathcal{Y} \to \mathcal{P}([U])$.

We know from previous work, see Table 2.4 (page 59) that

(1) such μ' can be represented by a preferential structure iff
$(\mu \subseteq)$ $\mu'(X) \subseteq X$
and
(μPR) $X \subseteq Y \Rightarrow \mu'(Y) \cap X \subseteq \mu'(X)$
hold (the structure can be chosen transitive),

(2) if $\mathcal{Y}$ is closed under finite unions, such μ' can be represented by a preferential structure iff, in addition
(μCUM) $\mu'(X) \subseteq Y \subseteq X \Rightarrow \mu'(X) = \mu'(Y)$
holds. (Again, the structure can be chosen transitive.)

We have to consider the translation of the conditions back from μ' to μ, where μ' is defined from μ as in Definition 3.4.3 (page 114).

$(\mu \subseteq)$: $\mu'([X]) \subseteq [X]$ does not necessarily entail $\mu(X) \subseteq X$, as, e.g., the possibility $\langle a,b\rangle \in X$, $\langle b,a\rangle \notin X$, but $\langle b,a\rangle \in \mu(X)$ is left open. But the converse is true, and natural, so we impose $\mu(X) \subseteq X$. Note, however, that this observation destroys full equivalence of the conditions for μ, we have to bear in mind that they only hold for μ'.

(μPR) for μ entails (μPR) for μ' :

Let $\mu'[X] := \mu'([X])$.

We have to show $[X] \subseteq [Y] \Rightarrow \mu'[Y] \cap [X] \subseteq \mu'[X]$.

Let $[X] \subseteq [Y]$. Consider X and Y such that $X \subseteq Y$. This is possible, e.g., add inverse pairs to Y as necessary, so by prerequisite $\mu(Y) \cap X \subseteq \mu(X)$. Note that we used here some prerequisite about domain closure of the original set $\mathcal{Y}$. Let $[a,b] \in \mu'[Y] \cap [X]$, so e.g. $\langle a,b\rangle \in \mu(Y) \subseteq Y$, and $\langle a,b\rangle \in X$ or $\langle b,a\rangle \in X$. In the latter case, $\langle b,a\rangle \in Y$, and by $(\mu S2)$ also $\langle b,a\rangle \in \mu(Y)$. So let without loss of generality $\langle a,b\rangle \in \mu(Y) \cap X \subseteq \mu(X)$, so $[a,b] \in \mu'[X]$.

(μCUM) for μ entails (μCUM) for μ' :

We have to show $\mu'[X] \subseteq [Y] \subseteq [X] \Rightarrow \mu'[X] = \mu'[Y]$.

Assume as above $\mu(X) \subseteq Y \subseteq X$ by $(\mu S2)$ and sufficient closure conditions of the original domain, so $\mu(X) = \mu(Y)$, and $\mu'[X] = \mu'[Y]$.

Additional properties

The following additional condition is very important, it says that x has minimal distance to itself, and all $[x,x]$ are minimal. It corresponds to a universal distance 0.

Definition 3.4.4

(μId) If there is for some x $\langle x,x\rangle \in A$, then $\mu(A) = \{\langle x,x\rangle \in A\}$

This condition translates directly to a condition about μ' :

If $[x,x] \in [A]$, then $\mu'[A] = \{[x,x] \in [A]\}$.

The following conditions are taken from [Sch04] (Section 3.2.3 there). The first two impose that 1 copy for each point suffice, in the finite and infinite case, the third imposes transitivity:

Definition 3.4.5

(1-fin) Let $X = A \cup B_1 \cup B_2$ and $A \cap \mu(X) = \emptyset$. Then $A \subseteq (A \cup B_1 - \mu(A \cup B_1)) \cup (A \cup B_2 - \mu(A \cup B_2))$.

(1-infin) Let $X = A \cup \bigcup\{B_i : i \in I\}$, and $A \cap \mu(X) = \emptyset$. Then $A \subseteq \bigcup\{A \cup B_i - \mu(A \cup B_i)\}$.

(T) $\mu(A \cup B) \subseteq A$, $\mu(B \cup C) \subseteq B \Rightarrow \mu(A \cup C) \subseteq A$.

It is also discussed there that in the 1-copy case, not all structures can be made transitive, contrary to the situation when we allow arbitrary many copies. The reader is referred there for further reference.

The limit version

We know from Section 2.3.2 (page 61), Fact 2.3.2 (page 63) that for *transitive* relations $\prec$ on $\mathcal{Y}$, the MISE system $\Lambda([X])$ has the following properties, corresponding to the system of sets of minimal elements:

(1) If $A \in \Lambda(Y)$, and $A \subseteq X \subseteq Y$, then $A \in \Lambda(X)$.

(2) If $A \in \Lambda(Y)$, and $A \subseteq X \subseteq Y$, and $B \in \Lambda(X)$, then $A \cap B \in \Lambda(Y)$.

(2a) Let $A \in \Lambda(Y)$, $A \subseteq X \subseteq Y$. Then, if $B \in \Lambda(Y)$, $A \cap B \in \Lambda(X)$. Conversely, if $B \in \Lambda(X)$, then $A \cap B \in \Lambda(Y)$.

(3) If $A \in \Lambda(Y)$, $B \in \Lambda(X)$, then there is $Z \subseteq A \cup B$ $Z \in \Lambda(Y \cup X)$.

Using $(\mu S2)$ and closure of the domain as above, we see that these conditions carry over to conditions about the MISE systems $\Lambda(X)$ of the original domain.

Higher order structures, and their limit version

We know from previous work, see Section 2.4.2 (page 72), Proposition 2.4.5 (page 84) and Proposition 2.4.7 (page 85) that any function μ' satisfying $(\mu \subseteq)$ can be represented by a higher preferential structure, and if μ' satisfies also (μCUM) (and $(\mu \bigcap)$), it can be represented by an essentially smooth higher preferential structure.

As this, too, was a very abstract result, independent of logic, it also carries over to our situation.

We discussed in Section 2.4.2.6 (page 85) that more research has to go into higher preferential structures before we can present an intuitively valid limit version. So this stays an open problem.

3.4.3.4 The original problem: products and projections

We know from previous work, see, e.g., [GS08f], Section 4.2.2.3, Example 4.2.4 (see also [Sch96-3], [Sch04], [GS08a]), that we need closure of the domain under finite unions for representation of cumulative functions by smooth structures. Obviously, the domain of products $X \times Y$ is usually not closed under finite unions. It is therefore hopeless to obtain nice representation results for many cases when we work directly with products, we have to work with more general sets of pairs, at least for testing the conditions. This was one of the reasons why we split the representation problem immediately into several layers. Recall also that we do not see the resulting sets of minimal elements, but only their projections - so we are quite "blind". We have shown in [Sch04] that such kind of blindness can have very serious consequences on representation results. See the results on absence of definability preservation there, in particular Section 5.2.3, Proposition 5.2.15 there, but also the "Hamster Wheels" described in [Sch04].

So let us see how we go from products to more arbitrary sets of pairs, and treat our problem with projections. We adopt the same strategy as we did in [Sch96-1]. (Our solution is more general, as we also manage more general input, to solve the smooth case.) In this article, we "hid" the problem in a formula $\phi(\langle a,b\rangle, A\times B)$, we make it now explicit. Obviously, our solution is not very nice, but we see no good alternative. To put this into perspective, the reader may want to look back to Section 2.5 (page 90).

A semantical revision function $\mid$, which assigns to every pair of (intuitively: model) sets $\langle A,B\rangle$ a subset $A\mid B\subseteq B$ is representable by a preferential, smooth, higher preferential, etc., structure iff the following condition $(\mu\exists)$ of existence of a more general function f holds:

Definition 3.4.6

$(\mu\exists)$ There is a representing function f for more arbitrary sets of pairs of elements $\langle a,b\rangle$ with the required properties, accepting all original product sets $A\times B$ as arguments, and such that the projection of the set of minimal elements on the second coordinate is $A\mid B$, i.e., $\pi_2(f(A\times B)) = A\mid B$.

Again, we pay generality with lack of elegance.

3.4.4 Semantic representation for generalized update and counterfactuals

3.4.4.1 Introduction

We work here in the spirit of the Stalnaker/Lewis semantics for counterfactual conditionals, see [Sta68], [Lew73]. We treat update and counterfactual conditionals together, as it has been tradition since the seminal paper by Katsuno and Mendelson, [KM90].

At first sight, everything seems simple, as we "simply add up what we see from the points on the left", i.e., $X\uparrow Y=\bigcup\{\{x\}\uparrow Y : x\in X\}=\bigcup\{\{y\in Y : \neg\exists y'\in Y(d(x,y')\prec d(x,y)\} : x\in X\}$. So it is tempting to do as if all distances from fixed x can be chosen independently (up to symmetry) for all other distances.

But things are a bit more complicated, as we "look into the pairs" $\langle x,y\rangle$, they all have to have x on the left. So situations about transitivity like $[ab]<[bc]<[cd]<[ad]$, thus $[ab]<[ad]$, an example discussed in [MS90] have to be considered. In this article, we constructed a common real valued distance from a set of independent distances (one for each point, showing the world as "it sees it"), by multiplying copies. The same strategy works again here.

In more detail:

- We first go from distances to equivalence classes respecting symmetry as we did in Section 3.4.3 (page 112).

- We then work with finite sequences of points like $\langle a, b, x, y \rangle$ and let essentially $d(\langle a, b, x \rangle, \langle a, b, x, y \rangle) := d(x, y)$, the latter the original distance, and make all other distances infinite (except when both sequences are identical, of course). Thus, we compare in a non-trivial way only sequences σ, τ, where τ is σ, with one element added at the end.

- In addition, we make all distances between shorter sequences bigger than all distances between longer sequences (unless they are already infinite or 0). This gives us a layered structure, repeating ω many times (in descending order) the distance structure between elements.

- Finally, we identify sequences with their end points, and have again that every point sees the world according to its own distances, but we work with a common abstract distance.

3.4.4.2 The general and smooth case

We describe now formally the idea.

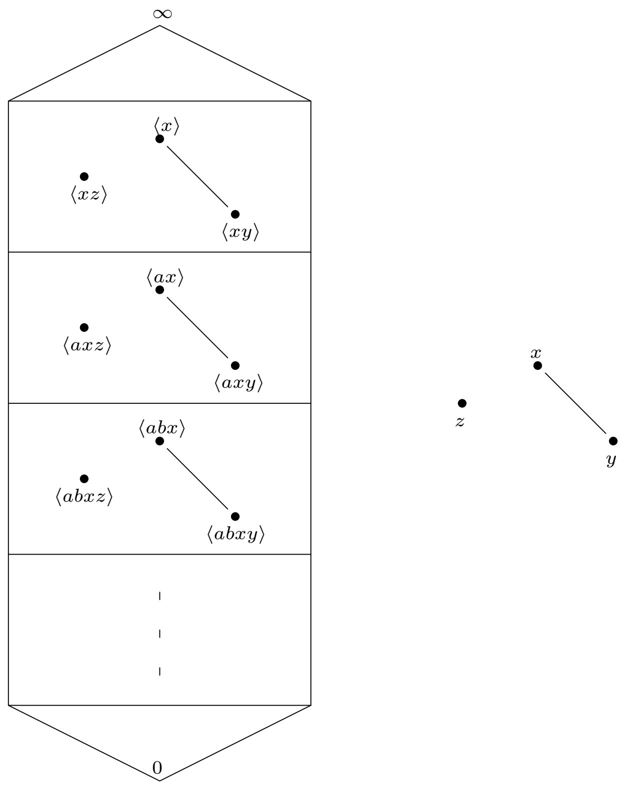
We consider sequences of points from U without direct repetitions, i.e., $\sigma_i \neq \sigma_{i+1}$. We then define

$$
d(\sigma, \tau) := \begin{cases}
0 & iff \quad \sigma = \tau \\[2ex]
d(\sigma_n, \sigma_{n+1}) & iff \quad \sigma = \sigma_0, .., \sigma_{n+1}, \tau = \sigma_0, .., \sigma_n \quad or \\
 & \qquad\;\; \tau = \tau_0, .., \tau_{n+1}, \sigma = \tau_0, .., \tau_n \\[2ex]
\infty & \qquad\;\; otherwise
\end{cases}
$$

This gives us distances between sequences of length $n+1$ and $n+2$. The distance between sequences is symmetrical iff the base distance is symmetrical.

Now, we have to arrange all these layers in a way to make distances between longer sequences smaller, resulting in the following picture, see Diagram 3.4.1 (page 118). In this diagram, the distance situation between points is shown on the right, between corresponding sequences on the left. 0 is the smallest distance, ∞ the biggest, and we have an infinite descending chain of layers. The distance between sequences is symmetrical iff the original distance was, it is transitive, iff the original distance was. (Formally, $d(\sigma, \sigma + x) > d(\tau, \tau + x)$ if σ and τ have the same end point, and σ is longer than τ is. Here, $+$ is appending one more element.)

Diagram 3.4.1



This is an $\mathcal{A}-$ranked structure, as discussed in [GS08d], see also [GS08f].

When we evaluate the structure, we are interested in the set of (original) points Y individually closest to some set of original points X. We look now at all sequences whose end points are in X, and from those to all sequences whose end points are in Y. The closest ones have the same end points as the (indiviually) closest to X elements of Y. So we are done, and can work with the new structure, representing it by some relation $\prec$.

Not yet, quite: We have destroyed cumulativity, if this was a property. But only superficially, as we argue now. *Inside* each layer, smoothness was preserved, but each layer gives the same answer, so, in the end, cumulativity *is* preserved.

More precisely, the property of cumulativity in our context is the following:

Definition 3.4.7

$(Up-Cum)$ Fix X, consider Y and Y'. If $Y' \subseteq Y$, and $X \uparrow Y \subseteq Y'$, then $X \uparrow Y = X \uparrow Y'$.

As the new structure was equivalent to the old structure in each layer, in each layer cumulativity is preserved, and the structure can thus be represented by a smooth relation in each layer, and we just put the layers together in the final construction, we obtain local smoothness. Details about cumulativity are left to the reader.

We now give a formal representation result for the basic case.

Consider the following property:

Definition 3.4.8

$(\mu\bigcup)$ If $\mathcal{X}$ is a cover of X, then $\mu(X) = \bigcup\{\mu(X') : X' \in \mathcal{X}\}$.

We postulate: $\uparrow$ satisfies $(\mu\bigcup)$ in the following sense on the left:

All $\mu_Y(X) := X \uparrow Y$ satisfy $(\mu\bigcup)$ for any Y.

We can now define:

Definition 3.4.9

(1) $x \Uparrow Y := \bigcap\{X \uparrow Y : x \in X\}$,

(2) $A \Uparrow B := \bigcup\{a \Uparrow B : a \in A\}$.

Note that $\{x\} \uparrow Y$ need not be defined, this way we avoid postulating that singletons are definable. We have:

Fact 3.4.3

Let the domain be closed under finite intersections, let $(\mu\bigcup)$ hold for $\uparrow$ on the left, then $A \Uparrow B = A \uparrow B$.

Proof

"$\subseteq$": $b \in A \Uparrow B \Rightarrow \exists a \in A.b \in a \Uparrow B \Rightarrow b \in A \uparrow B$, both implications by Definition 3.4.9 (page 120).

"$\supseteq$": $b \notin A \Uparrow B \Rightarrow$ (by definition) $\forall a \in A.b \notin a \Uparrow B \Rightarrow$ (by definition) $\forall a \in A \exists X(a \in X \wedge b \notin X \uparrow B) \Rightarrow$ (by $(\mu\bigcup)$) $\forall a \in A \exists X(a \in X \cap A \wedge b \notin (X \cap A) \uparrow B)$. As there is such $X \cap A$ for all $a \in A$, such $X \cap A$ form a cover of A, and by $(\mu\bigcup)$ again, $a \notin A \uparrow B$.

$\square$

Note that closure under finite intersections is here a very weak prerequisite, usually satisfied, and, in addition, it is *not* required on products or so, only on one coordinate.

We try to show now that relevant properties carry over from $\uparrow$ to $\Uparrow$. It seems, however, that we need an additional property:

Definition 3.4.10

(Approx) If $x \Uparrow Y \subseteq A$, then there is X such that $x \in X$, $X \uparrow Y \subseteq A$.

This condition is not as strong as requiring that singletons are in the domain, it says that we can approximate the results from singletons sufficiently well.

We have now:

Fact 3.4.4

$(\mu \subseteq)$ and (μPR) carry over from $\uparrow$ to $\Uparrow$, if (Approx) holds, the same is true for (μCUM).

More precisely:

(1) If $X \uparrow Y \subseteq Y$, then also $x \Uparrow Y \subseteq Y$ (for $x \in X$).

(2) If $Y \subseteq Y' \Rightarrow (X \uparrow Y') \cap Y \subseteq X \uparrow Y$, then also $Y \subseteq Y' \Rightarrow (x \Uparrow Y') \cap Y \subseteq x \Uparrow Y$.

(3) Let (Approx) and $(\cap)$ hold. Then: If $(X \uparrow Y') \subseteq Y \subseteq Y' \Rightarrow (X \uparrow Y) = (X \uparrow Y')$, then also $(x \Uparrow Y') \subseteq Y \subseteq Y' \Rightarrow (x \Uparrow Y) = (x \Uparrow Y')$.

(For all X, Y, Y', x, etc.)

Proof

(1) Trivial.

(2) Let $Y \subseteq Y'$. $y \in (x \Uparrow Y') \cap Y \Leftrightarrow y \in Y \wedge \forall X(x \in X \Rightarrow y \in X \uparrow Y') \Rightarrow$ (by (μPR) for $\uparrow$) $\forall X(x \in X \Rightarrow y \in X \uparrow Y) \Rightarrow y \in x \Uparrow Y$.

(3) Let $(x \Uparrow Y') \subseteq Y \subseteq Y'$. By (2) $(x \Uparrow Y') \cap Y \subseteq x \Uparrow Y$, by prerequisite $x \Uparrow Y' \subseteq Y$, so $x \Uparrow Y' \subseteq x \Uparrow Y$. It remains to show $x \Uparrow Y \subseteq x \Uparrow Y'$. Let $y \in x \Uparrow Y$, so for all X such that $x \in X$, $y \in X \uparrow Y$. Assume $y \notin x \Uparrow Y'$, so there is X', $x \in X'$, $y \notin X' \uparrow Y'$. We use now (Approx) to chose suitable X'' with $x \in X''$, such that $X'' \uparrow Y' \subseteq Y$. Now $x \in X' \cap X''$, so $y \in (X' \cap X'') \uparrow Y$. By $(\mu \bigcup)$ $(X' \cap X'') \uparrow Y' \subseteq X' \uparrow Y'$, so $y \notin (X' \cap X'') \uparrow Y'$. By $X'' \uparrow Y' \subseteq Y$ and $(\mu \bigcup)$ $(X' \cap X'') \uparrow Y' \subseteq Y$, so by (μCUM) for $\uparrow$, $(X' \cap X'') \uparrow Y = (X' \cap X'') \uparrow Y'$, contradiction.

□

We put our ideas together:

Let $\uparrow$ satisfy:

(1) $(\mu \bigcup)$ on the left,

(2) $(\mu \subseteq)$ and (μPR) on the right,

(3) let the domain be closed under $(\cap)$.

By Fact 3.4.4 (page 121), $x \Uparrow Y$ has suitable conditions, so we can represent $x \Uparrow Y$ individually for each x by a suitable relation. The construction of Section 3.4.4.2 (page 118) shows how to combine the individual relations. Fact 3.4.3 (page 120) shows that $X \Uparrow Y = X \uparrow Y$. So we can define $X \Uparrow Y$ from the individual $x \Uparrow Y$, having $X \uparrow Y = X \Uparrow Y$. This results in a suitable choice function for each element x, which can, individually, be represented by a relation $\prec_x$ (for each, fixed, x). The construction with ω many layers from the beginning of Section 3.4.4.2 (page 118) shows how to construct a global relation $\prec$ on a suitable space of sequences, whose evaluation gives the same results as the individual relations $\prec_x$. Conversely, any such order gives a choice function satisfying $(\mu \subseteq)$ and (μPR) on the right, and, in suitable interpretation, $(\mu \bigcup)$ on the left.

If $\uparrow$ satisfies (μCUM) on the right, and if we can approximate the results of singletons sufficiently well, i.e., if (Approx) holds, the choice functions $x \Uparrow (.)$ will also have the property (μCUM), they can represented by a smooth structure for each x, and the individual structures can be put together as above. The resulting global structure is not globally smooth, but locally smooth, which is sufficient. Conversely, if the structure is locally smooth, $\uparrow$ and $\Uparrow$ will satisfy (μCUM).

3.4.4.3 Further conditions and representation questions

- Note that we built the 0-property already into the basic construction, so there is nothing to do here.
- The 1-copy case was already discussed in Section 3.4.3.3 (page 115), we refer the reader there.
- For the limit version, all the important algebraic material stays valid, so we can use it to show that the same (unitary versions of) laws hold there as in the minimal variant, analogous to the TR case, see Section 3.4.3.3 (page 116).
- Higher preferential structures need not satisfy (μPR), so even distances not satisfying this condition can be represented by a suitable combination of techniques we are now familiar with.

3.4.5 Syntactic representation for generalized revision, update, counterfactuals

Our aim in this Section 3.4 (page 111) is not only to generalize the notions of distance, but also to generalize to what revision and update can be applied, and in which way (minimal or limit version). Consequently, the syntactic side is a less important part than in classical revision and update (and counterfactuals). So we will only indicate how to proceed, and will leave the rest to the reader - or future research.

First, some introductory remarks.

Translation from semantics to logics and back is discussed extensively in [Sch04], see, e.g., Section 3.4, 4.2.3 there. Particular attention is given to definability preservation and the problems arising from the lack of it, see there Chapter 5. Further discussion can be found in [GS08f], see, e.g., Section 5.4 there.

Discussion of the Theory Revision situation

We have to look back at Definition 3.4.6 (page 117).

The input, $A \times B$, is simple, and will be $M(\phi) \times M(\psi)$ for formulas, $M(S) \times M(T)$ for full theories.

The output will be (exactly - to obtain definability preservation) some $M(\psi')$ or $M(T')$.

Problems are in-between. We have to find a language (which will not just be classical propositional language) rich enough to

- Express finite unions of products of the type $M(\phi) \times M(\psi)$.
- Rich enough to describe the result of the choice function in the desired situation. Usually, this will not just be a product, or even a finite union of products.
- Going to the projection will probably not be very difficult.

It seems that one needs here in many cases specially tailored languages.

Discussion of the update/counterfactual situation

The case for update and counterfactuals is easier, essentially because we do not need any cross-comparison between $\langle a, b\rangle$ and $\langle a', b'\rangle$.

The conditions $(\mu \bigcup)$, $(\mu \subseteq)$, and (μPR) translate directly to the syntactical side - as usual. Care has to be taken to make the results of minimization definability preserving.

Example of syntactic conditions

To give a flavour of a full set of conditions, we quote from [Sch96-1] the logical counterpart of the semantical representation result, Theorem 3.1 there.

We first introduce a definition (Definition 3.3 there).

We use the abbreviation "cct" for "consistent complete theory", i.e., corresponding to a single model.

Fix a propositional language $\mathcal{L}$.

Definition 3.4.11

We consider two logics, $.^i : \mathcal{P}(\mathcal{L}) \times \mathcal{P}(\mathcal{L}) \to \mathcal{P}(\mathcal{L})$, $\langle S, T\rangle \mapsto \langle S, T\rangle^i \subseteq \mathcal{L}$.

We say that both logics $\langle S, T\rangle^i$ are given by a function $f : \boldsymbol{D} \times \boldsymbol{D} \to \mathcal{P}(M_{\mathcal{L}} \times M_{\mathcal{L}})$ iff for all theories S, T

$\langle S, T\rangle^i = \{\phi : \forall m \in \pi_i(f(M_S \times M_T)).m \models \phi\}$, where π_i is the projection on the $i-$th coordinate. Analogously, we say that they are given by a preferential structure on $M_{\mathcal{L}} \times M_{\mathcal{L}}$ iff f is given by such a structure.

We call a function f on pairs of models definability preserving (dp) iff for all theories S, T $\pi_1(f(M_S \times M_T)) = M_U$ and $\pi_2(f(M_S \times M_T)) = M_V$ for some theories U, V, where M_S is the set of $S-$models etc.

Note that then $\pi_i(f(M_S \times M_T)) = M_{\langle S,T\rangle^i}$.

For U, V complete, consistent theories (cct), S, T any theories, we abbreviate

$\psi(U, V, S, T) := \forall S', T'(U \vdash S' \vdash S \wedge V \vdash T' \vdash T \to U \vdash \langle S', T'\rangle^1 \wedge V \vdash \langle S', T'\rangle^2)$.

We quote now the syntactic result, Theorem 3.4 of [Sch96-1]:

Proposition 3.4.5

Let $\langle S,T\rangle^1$, $\langle S,T\rangle^2$ be two logics on pairs of theories. Then $\langle S,T\rangle^i$ are given by a dp preferential structure iff

(1) $\overline{S} = \overline{S'} \wedge \overline{T} = \overline{T'} \rightarrow \langle S,T\rangle^1 = \langle S',T'\rangle^1 \wedge \langle S,T\rangle^2 = \langle S',T'\rangle^2$

(2) $\langle S,T\rangle^i$ is classically closed

(3) $\langle S,T\rangle^1 \vdash S$, $\langle S,T\rangle^2 \vdash T$

(4S) If U is a cct with $U \vdash S$, then $U \vdash \langle S,T\rangle^1$ iff there is a cct V such that $V \vdash T$ and $\psi(U,V,S,T)$

(4T) If V is a cct with $V \vdash T$, then $V \vdash \langle S,T\rangle^2$ iff there is a cct U such that $U \vdash S$ and $\psi(U,V,S,T)$.

The proof is straightforward, and the reader is referred to [Sch96-1].

Chapter 4

Monotone and antitone semantic and syntactic interpolation

4.1 Introduction

4.1.1 Overview

The two chapters Chapter 4 (page 125) and Chapter 5 (page 165) are probably the core of the present book.

We treat very general interpolation problems for monotone and antitone, 2-valued and many-valued logics in the present chapter, splitting the question in two parts, "semantic interpolation" and "syntactic interpolation", show that the first problem, existence of semantic interpolation, has a simple and general answer, and reduce the second question, existence of syntactic interpolation to a definability problem. For the latter, we examine the concrete example of finite Goedel logics. We can also show that the semantic problem has two "universal" solutions, which depend only on one formula and the shared variables.

In Chapter 5 (page 165), we investigate three variants of semantic interpolation for non-monotonic logics, in syntactic shorthand of the types $\phi \mid\!\sim \alpha \vdash \psi$, $\phi \vdash \alpha \mid\!\sim \psi$, and $\phi \mid\!\sim \alpha \mid\!\sim \psi$, where α is the interpolant, and see that two variants are closely related to multiplication laws about abstract size, defining (or originating from) the non-monotonic logics. The syntactic problem is analogous to that of the monotonic case.

4.1.1.1 Background

Interpolation for classical logic is well-known, see [Cra57], and there are also non-classical logics for which interpolation has been shown, e.g., for Circumscription, see [Ami02]. An extensive overview of interpolation is found in [GM05]. Chapter 1 of this book gives a survey and a discussion and the chapter puts forward that interpolation can be viewed in many different ways and indeed 11 points of view of interpolation are discussed. The present text presents the semantic interpolation, this is a new 12th point of view.

4.1.2 Problem and Method

In classical logic, the problem of interpolation is to find for two formulas ϕ and ψ such that $\phi \vdash \psi$ a "simple" formula α such that $\phi \vdash \alpha \vdash \psi$. "Simple" is defined as: "expressed in the common language of ϕ and ψ".

Working on the semantic level has often advantages:

- results are robust under logically equivalent reformulations
- in many cases, the semantic level allows an easy reformulation as an algebraic problem, whose results can be generalized to other situations
- we can split a problem in two parts: a semantical problem, and the problem to find a syntactic counterpart (a definability problem)
- the semantics of many non-classical logics are built on relatively few basic notions like size, distance, etc., and we can thus make connections to other problems and logics
- in the case of preferential and similar logics, the very definition of the logic is already semantical (minimal models), so it is very natural to proceed on this level

This strategy - translate to the semantic level, do the main work there, and then translate back - has proved fruitful also in the present case.

Looking back at the classical interpolation problem, and translating it to the semantic level, it becomes: Given $M(\phi) \subseteq M(\psi)$ (the models sets of ϕ and ψ), is there a "simple" model set A such that $M(\phi) \subseteq A \subseteq M(\psi)$? Or, more generally, given model sets $X \subseteq Y$, is there "simple" A such that $X \subseteq A \subseteq Y$?

Of course, we have to define in a natural way, what "simple" means in our context. This is discussed below in Section 4.1.3.1 (page 127) .

Our separation of the semantic from the syntactic question pays immediately:

(1) We see that monotonic (and antitonic) logics *always* have a semantical interpolant. But this interpolant may not be definable syntactically.

(2) More precisely, we see that there is a whole interval of interpolants in above situation.

(3) We also see that our analysis generalizes immediately to many valued logics, with the same result (existence of an interval of interpolants).

(4) Thus, the question remains: under what conditions does a syntactic interpolant exist?

(5) In non-monotonic logics, our analysis reveals a deep connection between semantic interpolation and questions about (abstract) multiplication of (abstract) size.

4.1.3 Monotone and antitone semantic and syntactic interpolation

We consider here the semantic property of monotony or antitony, in the following sense (in the two-valued case, the generalization to the many-valued case is straightforward):

Let $\vdash$ be some logic such that $\phi \vdash \psi$ implies $M(\phi) \subseteq M(\psi)$ (the monotone case) or $M(\psi) \subseteq M(\phi)$ (the antitone case).

In the many-valued case, the corresponding property is that $\rightarrow$ (or $\vdash$) respects $\leq$, the order on the truth values.

4.1.3.1 Semantic interpolation

The problem (for simplicity, for the 2-valued case) reads now:

If $M(\phi) \subseteq M(\psi)$ (or, symmetrically $M(\psi) \subseteq M(\phi)$), is there a "simple" model set A, such that $M(\phi) \subseteq A \subseteq M(\psi)$, or $M(\psi) \subseteq A \subseteq M(\phi)$. Obviously, the problem is the same in both cases. We will see that such A will always exist, so all such logics have semantic interpolation (but not necessarily also syntactic interpolation).

The main conceptual problem is to define "simple model set". We have to look at the syntactic problem for guidance. Suppose ϕ is defined using propositional variables p and q, ψ using q and r. α has to be defined using only q. What are the models of α? By the very definition of validity in classical logic, neither p nor r have any influence on whether m is a model of α or not. Thus, if m is a model of α, we can modify m on p and r, and it will still be a model. Classical models are best seen as functions from the set of propositional variables to $\{true, false\}$, $\{t, f\}$, or so. In this terminology, any m with $m \models \alpha$ is "free" to choose the value for p and r, and we can write the model set A of α as $\{t, f\} \times M_q \times \{t, f\}$, where M_q is the set of values for q $\alpha-$models may have ($\emptyset$, $\{t\}$, $\{f\}$, $\{t, f\}$).

So, the semantic interpolation problem is to find sets which may be restricted on the common variables, but are simply the Cartesian product of the possible values for the other variables. To summarize: Let two model sets X and Y be given, where X itself is restricted on variables $\{p_1, \ldots, p_m\}$ (i.e. the Cartesian product for the rest), Y is restricted on $\{r_1, \ldots, r_n\}$, then we have to find a model set A which is restricted only on $\{p_1, \ldots, p_m\} \cap \{r_1, \ldots, r_n\}$, and such that $X \subseteq A \subseteq Y$, of course.

Formulated this way, our approach, the problem and its solution, has two trivial generalizations:

- for multi-valued logics we take the Cartesian product of more than just $\{t, f\}$.
- ϕ may be the hypothesis, and ψ the consequence, but also vice versa, there is no direction in the problem. Thus, any result for classical logic carries over to the core part of, e.g., preferential logics.

The main result for the situation with $X \subseteq Y$ is that there is always such a semantic interpolant A as described above (see Proposition 4.2.1 (page 131) for a simple case, and Proposition 4.2.3 (page 134) for the full picture). Our proof works also for "parallel interpolation", a concept introduced by Makinson et al., [KM07].

We explain and quote the latter result.

Suppose we have $f, g : M \rightarrow V$, where, intuitively, M is the set of all models, and V the set of all truth values. Thus, f and g give to each model a truth value, and, intuitively, f and g each code a model set, assigning to m TRUE iff m is in the model set, and FALSE iff not. We further assume that there is an order on the truth value set V. $\forall m \in M(f(m) \leq g(m))$ corresponds now to $M(\phi) \subseteq M(\psi)$, or $\phi \vdash \psi$ in classical logic. Each model m is itself a function from L, the set of propositional variables to V. Let now $J \subseteq L$. We say that f is insensitive to J iff the values of m on J are irrelevant: If $m \restriction (L-J) = m' \restriction (L-J)$, i.e., m and m' agree at least on all $p \in L-$J, then $f(m) = f(m')$. This corresponds to the situation where the variable p does not occur in the formula ϕ, then $M(\phi)$ is insensitive to p, as the value of any m on p does not matter for m being a model of ϕ, or not.

We need two more definitions:

Let $J' \subseteq L$, then $f^+(m_{J'}) := max\{f(m') : m' \restriction J' = m \restriction J'\}$ and $f^-(m_{J'}) := min\{f(m') : m' \restriction J' = m \restriction J'\}$.

We quote now Proposition 4.2.3 (page 134) , slightly simplified:

Proposition 4.1.1

Let $f, g : M \to V$, $f(m) \leq g(m)$ for all $m \in M$. Let $L = J \cup J' \cup J''$, let f be insensitive to J, g be insensitive to J''.

Then $f^+(m_{J'}) \leq g^-(m_{J'})$ for all $m_{J'} \in M \restriction J'$, and any $h : M \restriction J' \to V$ which is insensitive to $J \cup J''$ is an interpolant iff

$f^+(m_{J'}) \leq h(m_J m_{J'} m_{J''}) = h(m_{J'}) \leq g^-(m_{J'})$ for all $m_{J'} \in M \restriction J'$.

(h can be extended to the full M in a unique way, as it is insensitive to $J \cup J''$.)

See Diagram 4.2.1 (page 134).

4.1.3.2 The interval of interpolants

Our result has an additional reading: it defines an interval of interpolants, with lower bound $f^+(m_{J'})$ and upper bound $g^-(m_{J'})$. But these interpolants have a particular form. If they exist, i.e. iff $f \leq g$, then $f^+(m_{J'})$ depends only on f and J' (and m), but *not* on g, $g^-(m_{J'})$ only on g and J', *not* on f. Thus, they are universal, as we have to look only at one function and the set of common variables.

Moreover, we will see in Section 4.3.3 (page 137) that they correspond to simple operations on the normal forms in classical logic. This is not surprising, as we "simplify" potentially complicated model sets by replacing some coordinates with simple products. The question is, whether our logic allows to express this simplification, classical logic does.

4.1.3.3 Syntactic interpolation

Recall the problem described at the beginning of Section 4.1.3.1 (page 127) . We were given $M(\phi) \subseteq M(\psi)$, and were looking for a "simple" model set A such that $M(\phi) \subseteq A \subseteq M(\psi)$. We just saw that such A's exists, and were able to describe an interval of such A's. But we have no guarantee that any such A is definable, i.e., that there is some α with $A = M(\alpha)$.

In classical logic, such α exists, see, e.g., Proposition 4.4.1 (page 147)), but also Section 4.3.3 (page 137). Basically, in classical logic, $f^+(m_{J'})$ and $g^-(m_{J'})$ correspond to simplifications of the formulas expressed in normal form, see Fact 4.3.3 (page 140) (in a different notation, which we will explain in a moment). This is not necessarily true in other logics, see Example 4.4.1 (page 157).

We find here again the importance of definability preservation, a concept introduced by one of us in [Sch92].

If we have projections (simplifications), see Section 4.3 (page 136), we also have syntactic interpolation. At present, we do not know whether this is a necessary condition for all natural operators.

We can also turn the problem around, and just define suitable operators. This is done in Section 4.3.3 (page 137), Definition 4.3.2 (page 138) and Definition 4.3.3 (page 138) . There is a slight problem, as one of the operands is a *set* of propositional variables, and not a formula, as usual. One, but certainly not the only one, possibility is to take a formula (or the corresponding model set) and "extract" the "relevant" variables from it, i.e., those, which cannot be replaced by a product. Assume now that f is one of the generalized model "sets", then:

Given f, define

(1) $(f \uparrow J)(m) := sup\{f(m') : m' \in M,\ m \upharpoonright J = m' \upharpoonright J\}$

(2) $(f \downarrow J)(m) := inf\{f(m') : m' \in M,\ m \upharpoonright J = m' \upharpoonright J\}$

(3) $\phi!\psi$ by:
$f_{\phi!\psi} := f_\phi \uparrow (L - R(\psi))$

(4) $\phi?\psi$ by:
$f_{\phi?\psi} := f_\phi \downarrow (L - R(\psi))$

We then obtain for classical logic (see Fact 4.3.3 (page 140)):

Fact 4.1.2

Let $J := \{p_{1,1}, \ldots, p_{1,m_1}, \ldots, p_{n,1}, \ldots, p_{n,m_n}\}$

(1) Let $\phi_i := \pm p_{i,1} \wedge \ldots \wedge \pm p_{i,m_i}$ and $\psi_i := \pm q_{i,1} \wedge \ldots \wedge \pm q_{i,k_i}$, let $\phi := (\phi_1 \wedge \psi_1) \vee \ldots \vee (\phi_n \wedge \psi_n)$. Then $\phi \uparrow J = \phi_1 \vee \ldots \vee \phi_n$.

(2) Let $\phi_i := \pm p_{i,1} \vee \ldots \vee \pm p_{i,m_i}$ and $\psi_i := \pm q_{i,1} \vee \ldots \vee \pm q_{i,k_i}$, let $\phi := (\phi_1 \vee \psi_1) \wedge \ldots \wedge (\phi_n \vee \psi_n)$. Then $\phi \downarrow J = \phi_1 \wedge \ldots \wedge \phi_n$.

In a way, these operators are natural, as they simplify definable model sets, so they can be used as a criterion of the expressive strength of a language and logic: If X is definable, and Y is in some reasonable sense simpler than X, then Y should also be definable. If the language is not sufficiently strong, then we can introduce these operators, and have also syntactic interpolation.

4.1.3.4 Finite Goedel logics

The semantics of finite (intuitionistic) Goedel logics is a finite chain of worlds, which can also be expressed by a totally ordered set of truth values 0 . . . n (see Section 4.4.3 (page 148)). Let FALSE and TRUE be the minimal and maximal truth values. ϕ has value false, iff it holds nowhere, and TRUE, iff it holds everywhere, it has value 1 iff it holds from world 2 onward, etc. The operators are classical $\wedge$ and $\vee$, negation $\neg$ is defined by $\neg(FALSE) = TRUE$ and $\neg(x) = FALSE$ otherwise. Implication $\rightarrow$ is defined by $\phi \rightarrow \psi$ is TRUE iff $\phi \leq \psi$ (as truth values), and the value of ψ otherwise.

More precisely, where f_ϕ is the model value function of the formula ϕ :

negation $\neg$ is defined by:

$$f_{\neg\phi}(m) := \begin{cases} TRUE \quad iff \quad f_\phi(m) = FALSE \\ \\ FALSE \qquad\qquad otherwise \end{cases}$$

implication $\rightarrow$ is defined by:

$$f_{\phi\rightarrow\psi}(m) := \begin{cases} TRUE \quad iff \quad f_\phi(m) \leq f_\psi(m) \\ \\ f_\psi(m) \qquad\qquad otherwise \end{cases}$$

see Definition 4.4.2 (page 148) in Section 4.4.3 (page 148). We show in Section 4.4.3.2 (page 157) the well-known result that such logics for 3 worlds (and thus 4 truth values) have no interpolation, whereas the corresponding logic for 2 worlds has interpolation. For the latter logic, we can still find a kind of normal form, though $\rightarrow$ cannot always be reduced. At least we can avoid nested implications, which is not possible in the logic for 3 worlds.

We also discuss several "hand made" additional operators which allow us to define sufficiently many model sets to have syntactical interpolation - of course, we *know* that we have semantical interpolation. A more systematic approach was discussed above, the operators $\phi!\psi$ and $\phi?\psi$.

4.2 Monotone and antitone semantic interpolation

We explain what is happening here. Assume $A(p,q)$ proves $B(q,r)$ where the set of models $M(A(p,q))$ is a subset of $M(B(q,r))$ *or vice versa*, i.e., B proves A and the subset relation is also the inverse. We discuss here the first variant. - The common language is p. This means for all p,q,r ($A(p,q)$ proves $B(q,r)$) or equivalently for all q(for some p A) proves (for all r B)). Semantically this means, in the monotonic case, the (union on p models of A) is a subset of (intersection on r models of B). We want to extract from this a set of models of q interpolating in between. This is what the set theoretical manipulation below does. The result is formulated in Proposition 4.2.1 (page 131).

Once we find the semantic interpolant we ask under what conditions can we find a syntactic C to do the job. This we investigate in the rest of the section. In classical logic, the semantic result carries over immediately to the syntactic level, as is shown in Proposition 4.4.1 (page 147). C can be found in many cases by enriching the language. There are papers in the literature with a title "repairing interpolation for logic X", e.g., by Areces, Blackburn, and Marx, see [ABM03], this is what they do for some particular logic X.

Thus, the following interpolation results can be read upward (monotonic logic) or downward (the core of non-monotonic logic, in the following sense: γ is the theory of the minimal models of α, and not just any formula which holds in the set of minimal models - which would be downward, and then upward again in the sense of model set inclusion), in the latter case we have to be careful: we usually cannot go upward again, so we have the sharpest possible case in mind. The case of mixed movement - down and then up - as in full non-monotonic logic is treated in Section 5.3 (page 193).

As a warming up exercise, we do first a simplified version of the two-valued case, giving only the lower bound. Parts (2) and (3) of the following Proposition concern "parallel interpolation", see [KM07].

4.2.1 The two-valued case

Recall that we can work here with sets of models, which are named Σ etc., to remind us that model sets are sets of sequences. Part (2) and (3) concern "parallel interpolation", a terminology used by Makinson et al. in [KM07]. The proofs of these parts are straightforward generalizations of the simple case, they are mentioned for completeness' sake.

Proposition 4.2.1

Let $\Sigma' \subseteq \Sigma \subseteq \Pi$, where $\Pi = \Pi\{X_i : i \in L\}$.

Recall Definition 2.2.5 (page 43) for the definitions of I and R.

(1) Let $\Sigma'' := \Sigma' \upharpoonright (R(\Sigma) \cap R(\Sigma')) \times \Pi \upharpoonright (I(\Sigma) \cup I(\Sigma'))$.

Then $\Sigma' \subseteq \Sigma'' \subseteq \Sigma$.

The following two results concern "parallel interpolation", terminology introduced by D.Makinson in [KM07]. Thus in the first case, Σ' is a product, in the second case, Σ is a product. We do interpolation for a whole family of partial lower or upper bounds in parallel, thus its name.

(2) Let $\mathcal{J}$ be a disjoint cover of L.

Let $\Sigma' = \Pi\{\Sigma'_K : K \in \mathcal{J}\}$ with $\Sigma'_K \subseteq \Pi\{X_i : i \in K\}$.

Let $\Sigma''_K := \Sigma'_K \upharpoonright (R(\Sigma) \cap R(\Sigma'_K)) \times \Pi\{X_i : i \in K, i \in I(\Sigma) \cup I(\Sigma'_K)\}$.

Let $\Sigma'' := \Pi\{\Sigma''_K : K \in \mathcal{J}\}$ (re-ordered).

Then $\Sigma' \subseteq \Sigma'' \subseteq \Sigma$.

(3) Let $\mathcal{J}$ be a disjoint cover of L.

Let $\Sigma = \Pi\{\Sigma_K : K \in \mathcal{J}\}$ with $\Sigma_K \subseteq \Pi\{X_i : i \in K\}$.

Let $\Sigma''_K := \Sigma' \upharpoonright (R(\Sigma') \cap R(\Sigma_K)) \times \Pi\{X_i : i \in K, i \in I(\Sigma_K) \cup I(\Sigma')\}$.

Let $\Sigma'' := \Pi\{\Sigma''_K : K \in \mathcal{J}\}$ (re-ordered).

Then $\Sigma' \subseteq \Sigma'' \subseteq \Sigma$.

Proof

(1)

(1.1) $\Sigma' \subseteq \Sigma''$ is trivial.

(1.2) $\Sigma'' \subseteq \Sigma$:

We can use Fact 2.2.2 (page 43) (2), or argue directly. We will do the latter.

Let $m \in \Sigma''$, so, by definition, there is $m' \in \Sigma'$ such that $m \upharpoonright (R(\Sigma) \cap R(\Sigma')) = m' \upharpoonright (R(\Sigma) \cap R(\Sigma'))$. Define m'' as follows: On $(R(\Sigma) \cap R(\Sigma')) \cup I(\Sigma')$, m'' is like m, on the other i, m'' is like m'. m' differs from m'' at most on $I(\Sigma')$, so by definition of $I(\Sigma')$, $m'' \in \Sigma' \subseteq \Sigma$. m'' is like m at least on $(R(\Sigma) \cap R(\Sigma')) \cup I(\Sigma') \supseteq R(\Sigma)$, so by definition of $R(\Sigma)$, $m \in \Sigma$.

(2)

(2.1) $\Sigma' \subseteq \Sigma''$.

$\Sigma'_K \subseteq \Sigma''_K$, so by $\Sigma'' = \Pi\Sigma''_K$ the result follows.

(2.2) $\Sigma'' \subseteq \Sigma$.

Let $m'' \in \Sigma''$, $m'' = \circ\{m''_K : K \in \mathcal{J}\}$ for suitable $m''_K \in \Sigma''_K$, where $\circ$ stands for the composition (or concatenation) of partial sequences or models. Consider m''_K. By definition of Σ''_K, there is $m'_K \in \Sigma'_K$ s.t. $m''_K \upharpoonright (R(\Sigma'_K) \cap R(\Sigma)) = m'_K \upharpoonright (R(\Sigma'_K) \cap R(\Sigma))$, so there is $n'_K \in \Sigma'_K$ s.t. $m''_K \upharpoonright R(\Sigma) = n'_K \upharpoonright R(\Sigma)$. Let $n' := \circ\{n'_K : K \in \mathcal{J}\}$, so by $\Sigma' = \Pi\{\Sigma'_K : K \in \mathcal{J}\}$ $n' \in \Sigma' \subseteq \Sigma$. But $m'' \upharpoonright R(\Sigma) = n' \upharpoonright R(\Sigma)$, so $m'' \in \Sigma$.

(3)

We first show $R(\Sigma_K) = R(\Sigma) \cap K$.

Let $i \in I(\Sigma_K)$, then $\Sigma_K = \Sigma_K \upharpoonright (K - \{i\}) \times X_i$, but $\Sigma = \Pi\{\Sigma_K : K \in \mathcal{J}\}$, so $\Sigma = \Sigma \upharpoonright (L - \{i\}) \times X_i$, and $i \in I(\Sigma)$. Conversely, let $i \in I(\Sigma) \cap K$, then $\Sigma = \Sigma \upharpoonright (L - \{i\}) \times X_i$, so $\Sigma \upharpoonright K = \Sigma \upharpoonright (K - \{i\}) \times X_i$, so $i \in I(\Sigma_K)$.

(3.1) $\Sigma' \subseteq \Sigma''$.

$\Sigma' \upharpoonright K \subseteq \Sigma''_K$, so by $\Sigma'' = \Pi\{\Sigma''_K : K \in \mathcal{J}\}$, $\Sigma' \subseteq \Sigma''$.

(3.2) $\Sigma'' \subseteq \Sigma$.

By $\Sigma = \Pi\{\Sigma_K : K \in \mathcal{J}\}$, it suffices to show $\Sigma''_K \subseteq \Sigma_K$.

Let $m''_K \in \Sigma''_K$. So there is $m' \in \Sigma'$ s.t. $m' \upharpoonright (R(\Sigma_K) \cap R(\Sigma')) = m''_K \upharpoonright (R(\Sigma_K) \cap R(\Sigma'))$, so there is $n' \in \Sigma' \subseteq \Sigma$ s.t. $n' \upharpoonright R(\Sigma_K) = m''_K \upharpoonright R(\Sigma_K)$, so by $R(\Sigma_K) = R(\Sigma) \cap K$, there is $m \in \Sigma$ s.t. $m \upharpoonright K = m''_K$, so $m''_K \in \Sigma_K$.

□

4.2.2 The many-valued case

For the basic definitions and facts, see Section 2.2.2 (page 47).

We assume here that the max and min of arbitrary sets of truth values will always exist. In Section 5.5 (page 209), we consider argumentation, and see the set of arguments for some formula ϕ as its truth value. There, only the max (i.e., union) will always exist, we will see there that this is also sufficient for some form of interpolation.

Recall that we do not work with sets of models or sequences any more, but with arbitrary functions $f, g, h : M \to V$, where each $m \in M$ is a function $m : L \to V$, where, intuitively, L stands for the set of propositional variables, V for the set of truth values, M is the set of many-valued models, and f etc. are functions (intuitively, $f = f_\phi$ etc.) assigning each m a value, intuitively, the value of ϕ in m. Again, we will consider $f \leq g$ and look for some h with $f \leq h \leq g$, where $I(f) \cup I(g) \subseteq I(h)$.

Definition 4.2.1

(1) Let $J \subseteq L$, $f : M \to V$. Define

$f^+(m \upharpoonright J) := max\{f(m') : m \upharpoonright J = m' \upharpoonright J\}$ and

$f^-(m \upharpoonright J) := min\{f(m') : m \upharpoonright J = m' \upharpoonright J\}$.

(Similarly, if m is defined only on J, the condition is $m' \upharpoonright J = m$, instead of $m \upharpoonright J = m' \upharpoonright J$.)

(2) Call M rich iff for all $m, m' \in M$, $J \subseteq L$, $(m \upharpoonright J) \cup (m' \upharpoonright (L - J)) \in M$. (I.e., we may cut and paste models.)

This assumption is usually given, it is mainly here to remind the reader that it is not trivial, and we have to make sure it really holds. A problem might, e.g., arise when we consider only subsets of all models, i.e., some $M' \subseteq M$, and not the full M.

Note that the possibility of arbitrary combinations of models is also an aspect of independence.

(3) A reminder: Call $f : M \to V$ insensitive to $J \subseteq L$ iff for all $m, n : m \upharpoonright (L - J) = n \upharpoonright (L - J)$ implies $f(m) = f(n)$ - i.e., the values of m on J have no importance for f. See Section 2.2.2 (page 47) , Table 2.3 (page 54).

(4) We will sometimes write m_J for $m \upharpoonright J$.

Remark 4.2.2

Let $J \subseteq L$, $m \in M$, where $m : L \to V$, and $f : M \to V$.

(1) Obviously, if $J = L$, then $f^+(m_J) = f^-(m_J) = f(m)$.

(2) We define the following ternary operators + and -:
$+(f, m, J) := f^+(m_J) \in V$, $-(f, m, J) := f^-(m_J) \in V$.

(3) Usually, we fix f and J, and are interested in the value $f^+(m_J)$ and $f^-(m_J)$ for various m. Seen this way, we have binary operators
$+(f, J) : M \to V$, defined by $+(f, J)(m) := f^+(m_J)$ and
$-(f, J) : M \to V$, defined by $-(f, J)(m) := f^-(m_J)$,
i.e., the results $+(f, J)$ and $-(f, J)$ are new model functions like f again.

(4) When we fix now J, we have unary functions $+(f)$ and $-(f)$ which assign to the old functions f new functions f^+ and f^-. This is probably the best way to consider our new operators.

By definition, $f^-(m) \leq f(m) \leq f^+(m)$ for all m.

(5) In the two-valued case, f is a model set X, and for any model m, $m \in X$ or $m \notin X$. Fix now J.

In above sense, $m' \in X^+$ iff there is m such that $m \upharpoonright J = m' \upharpoonright J$ and $m \in X$, and $m' \in X^-$ iff for all m such that $m \upharpoonright J = m' \upharpoonright J$, also $m \in X$ holds.

Thus, $X^- \subseteq X \subseteq X^+$.

(6) When we thus fix J, + and - are new operators on model set functions, or on model sets in the two-valued case. As such, they are similar to other new operators, like the $\mu-$operator of preferential structures. But they are simpler, in the following sense: they do not require an additional structure like a relation $\prec$, but are "built into" the model structure itself, as they only need the algebraic structure already present in products. Thus, in a certain way, they are more elementary.

But, just as μ need not preserve definability, neither do + or -; if $f = f_\phi$ for some ϕ, then there need not be ψ such that $f^+ = f_\psi$ or $f^- = f_\psi$.

Let $L = J \cup J' \cup J''$ be a disjoint union. If $f : M \to V$ is insensitive to $J \cup J''$, we can define for $m_{J'} : J' \to V$ $f(m_{J'})$ as any $f(m')$ such that $m' \upharpoonright J' = m_{J'}$.

Proposition 4.2.3

Let M be rich, $f, g : M \to V$, $f(m) \leq g(m)$ for all $m \in M$. Let $L = J \cup J' \cup J''$, let f be insensitive to J, g be insensitive to J''.

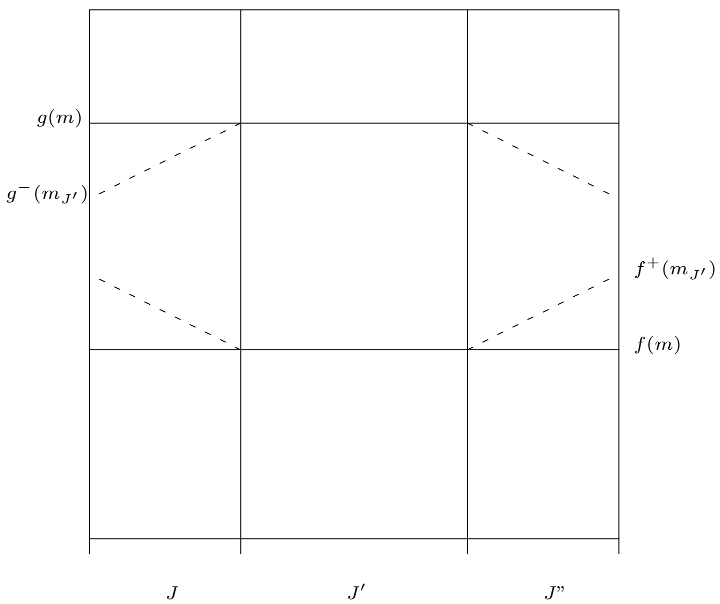
Then $f^+(m_{J'}) \leq g^-(m_{J'})$ for all $m_{J'} \in M \upharpoonright J'$, and any $h : M \upharpoonright J' \to V$ which is insensitive to $J \cup J''$ is an interpolant iff

$f^+(m_{J'}) \leq h(m_J m_{J'} m_{J''}) = h(m_{J'}) \leq g^-(m_{J'})$ for all $m_{J'} \in M \upharpoonright J'$.

(h can be extended to the full M in a unique way, as it is insensitive to $J \cup J''$, so it does not really matter whether we define h on L or on J'.)

See Diagram 4.2.1 (page 134).

Diagram 4.2.1



Semantic interpolation, $f \leq g$

Proof

Let $L = J \cup J' \cup J''$ be a pairwise disjoint union. Let f be insensitive to J, g be insensitive to J''.

$h : M \to V$ will have to be insensitive to $J \cup J''$, so we will have to define h on $M \restriction J'$, the extension to M is then trivial.

Fix arbitrary $m_{J'} : J' \to V$, $m_{J'} = m \restriction J'$ for some $m \in M$. We first show $f^+(m_{J'}) \leq g^-(m_{J'})$.

Proof: Choose $m_{J''}$ such that $f^+(m_{J'}) = f(m_J m_{J'} m_{J''})$ for any m_J. (Recall that f is insensitive to J.) Let $n_{J''}$ be one such $m_{J''}$. Likewise, choose m_J such that $g^-(m_{J'}) = g(m_J m_{J'} m_{J''})$ for any $m_{J''}$. Let n_J be one such m_J. Consider $n_J m_{J'} n_{J''} \in M$ (recall that M is rich). By definition, $f^+(m_{J'}) = f(n_J m_{J'} n_{J''})$ and $g^-(m_{J'}) = g(n_J m_{J'} n_{J''})$, but by prerequisite $f(n_J m_{J'} n_{J''}) \leq g(n_J m_{J'} n_{J''})$, so $f^+(m_{J'}) \leq g^-(m_{J'})$.

Thus, any h such that h is insensitive to $J \cup J''$ and

(Int) $f^+(m_{J'}) \leq h(m) := h(m_{J'}) \leq g^-(m_{J'})$

is an interpolant for f and g. The definition $h(m) := h(m_{J'})$ is possible, as h is insensitive to $J \cup J''$.

$f(m_J m_{J'} m_{J''}) \leq h(m_J m_{J'} m_{J''}) \leq g(m_J m_{J'} m_{J''})$ follows trivially, using above notation: $f(m_J m_{J'} m_{J''}) \leq f(m_J m_{J'} n_{J''}) = f^+(m_{J'}) \leq h(m_{J'}) = h(m_J m_{J'} m_{J''}) \leq g^-(m_{J'}) = g(n_J m_{J'} m_{J''}) \leq g(m_J m_{J'} m_{J''})$.

But (Int) is also a necessary condition.

Proof:

Suppose h is insensitive to $J \cup J''$ and $h(m_J m_{J'} m_{J''}) = h(m_{J'}) < f^+(m_{J'})$. Let $n_{J''}$ be as above, i.e., $f(m_J m_{J'} n_{J''}) = f^+(m_{J'})$ for any m_J. Then $h(m_J m_{J'} n_{J''}) = h(m_{J'}) < f^+(m_{J'}) = f(m_J m_{J'} n_{J''})$, so h is not an interpolant.

The proof that $h(m_J m_{J'} m_{J''}) = h(m_{J'})$ has to be $\leq g^-(m_{J'})$ is analogous.

We summarize:

f and g have an interpolant h, and h is an interpolant for f and g iff h is insensitive to $J \cup J''$ and for any $m_{J'} \in M \restriction J'$ $f^+(m_{J'}) \leq h(m_J m_{J'} m_{J''}) = h(m_{J'}) \leq g^-(m_{J'})$.

□

Definition 4.2.2

It is thus justified to call in above situation

$f^+(m_{J'})$ and $g^-(m_{J'})$ the standard interpolants,

more precisely, we call h such that $h(m) = f^+(m_{J'})$ or $h(m) = g^-(m_{J'})$ with $R(h) \subseteq J'$ a standard interpolant.

It seems that the same technique can be used to show many-valued semantic interpolation for modal and first-order logic, but we have not checked in detail.

4.3 The interval of interpolants in monotonic or antitonic logics

4.3.1 Introduction

We take now a closer look at the interval of interpolants, with already some remarks on the syntactic side.

By Proposition 4.2.3 (page 134), we have an interval of interpolants, and the extremes, the standard interpolants, see Definition 4.2.2 (page 135), are particularly interesting, for the following reasons:

(1) precisely because they are the extremes

(2) they are universal in the sense that they depend only one of the two functions and the variable set, more precisely:

Definition 4.3.1

Let $L = J \cup J' \cup J''$, let $f : M \to V$ be insensitive to J.

(1) h is an upper universal interpolant for f and J' iff for all g such that g is insensitive to J'', and $f \leq g$, h is an interpolant for f and g, i.e., $f \leq h \leq g$ and h is insensitive to $J \cup J''$.

(2) h is a lower universal interpolant for f and J' iff for all g such that g is insensitive to J'', and $g \leq f$, h is an interpolant for f and g, i.e., $g \leq h \leq f$ and h is insensitive to $J \cup J''$.

So $f^+(m_{J'})$ is (the only - provided we have sufficiently many g to consider) upper for f, $g^-(m_{J'})$ the (again, only) lower interpolant for g.

(3) They have a particularly simple structure. E.g., we could also choose for one $m_{J'}$ the lower boudary, for another $m'_{J'}$ the upper boundary, etc., and compose an interpolant in this somewhat arbitrary way.

(4) The operators + and - are themselves interesting and elementary.

(5) It might be difficult to find necessary and sufficient criteria for the existence of some syntactic interpolant in the interval, whereas it might be possible to find such criteria for the existence of one or both of the universal interpolants.

(6) In particular, we can ask the following questions:

If f is given by some ϕ, i.e., $f = f_\phi$, and $J' \subseteq L$ as above, defining $f^+_\phi(m) := f^+_\phi(m_{J'})$ and $f^-_\phi(m) := f^-_\phi(m_{J'})$, can we find formulas ψ and ψ' containing only variables from J' such that $f_\psi = f^+_\phi$ and $f_{\psi'} = f^-_\phi$? In more detail:

- Is this possible in classical logic?
- Is this possible in other logics?
- Do we find criteria for the language (operators, truth values, etc.) which guarantee that such ψ and ψ' exist?

We turn to some examples and a simple fact.

4.3.2 Examples and a simple fact

Example 4.3.1

This trivial, many-valued example shows that infimum and supremum might coincide. Take 3 truth values, $0, 1, 2$, in this order, and 3 propositional variables, p, q, r. We want to fix q, and let p and r float. $\phi := p \wedge q \wedge r$, $\phi' := p \vee q \vee r$, so $\phi \leq \phi'$. We fix q at 1, so $sup\{F_\phi(m) : m(q) = 1\} = inf\{F_{\phi'}(m) : m(q) = 1\} = 1$.

We present now two examples for the interval of interpolants.

Example 4.3.2

Consider classical logic with 4 propositional variables, p, q, r, s.

Let $\phi := p \wedge q \wedge r$, $\psi := q \vee r \vee s$. Obviously, $\phi \models \psi$.

Let $f := f_\phi$ assign to a model m the truth value ϕ has in m, likewise for $g := f_\psi$ for ψ.

Let $f'(m) := sup\{f(m') : m \upharpoonright \{q, r\} = m' \upharpoonright \{q, r\}\}$, $g'(m) := inf\{g(m') : m \upharpoonright \{q, r\} = m' \upharpoonright \{q, r\}\}$, then $f'(m) = f_{q \wedge r}(m)$, $g'(m) = f_{q \vee r}(m)$.

f' and g' are the bounds of the interval, and, e.g., q and r are both inside the interval, we have $f' \leq f_q, f_r \leq g'$.

Example 4.3.3

(See Section 4.4.3 (page 148) for motivation.)

Consider a finite intuitionistic Goedel language over p, q, r, with two worlds, and an additional operator $E\phi$, which says that ϕ holds everywhere. We have 3 truth values, $0 < 1 < 2$, $\wedge$ and $\vee$ with the usual interpretation of inf and sup, intuitionistic negation with $\neg\phi$ has value 2 iff ϕ has value 0, and 0 otherwise, so $\neg\neg\phi$ has value 0 iff ϕ has value 0, and 2 otherwise, and $E\phi$ has value 2 iff ϕ has value 2, and 0 otherwise.

Consider $\phi := p \wedge Eq$ and $\psi := \neg\neg q \vee r$. Obviously, $\phi \models \psi$.

$f := f_\phi$ assigns to a model m the truth value ϕ has in m, likewise for $g := f_\psi$ for ψ.

Let $f'(m) := sup\{f(m') : m \upharpoonright \{q\} = m' \upharpoonright \{q\}\}$, $g'(m) := inf\{g(m') : m \upharpoonright \{q\} = m' \upharpoonright \{q\}\}$. It suffices to consider the truth values for q : If $m(q) = 0$, then $f'(m) = g'(m) = 0$, if $m(q) = 1$, then $f'(m) = 0$, $g'(m) = 2$, If $m(q) = 2$, then $f'(m) = g'(m) = 2$.

So $f'(m) = f_{Eq}(m)$, $g'(m) = f_{\neg\neg q}(m)$.

f' and g' are the bounds of the interval, and, q is inside the interval, we have $f' \leq f_q \leq g'$.

4.3.3 The analoga of + and - as new semantic and syntactic operators

4.3.3.1 Motivation

We introduced and discussed the operators + and - in Definition 4.2.1 (page 132) and Remark 4.2.2 (page 133). In the binary version, see Remark 4.2.2 (page 133), they were $+(f, J)$ and $-(f, J)$, where f may be the model function of some formula ϕ, $f = f_\phi$. So, intuitively, we can read $+(\phi, J)$ and $-(\phi, J)$ as new syntactic operators.

A usual binary operator between formulas has two formulas as arguments. J, however, is just a set of propositional variables. Of course, we may take the conjunction of J in the finite case. As formulas are finite, most variables are "free" anyway, so finiteness is not a real restriction. Still, so far, one of the formulas has to be a conjunction of variables, which is bizarr for a normal operator. We have to generalize, but are relatively free how we do this, as we really just need the set of variables. The perhaps simplest way to do this is to take $R(\psi)$, the set of relevant variables of ψ. Then $+(\phi, \psi) := f_\phi^+(m \restriction R(\psi))$ and $-(\phi, \psi) := f_\phi^-(m \restriction R(\psi))$ look more respectable. Note, however, that, e.g., $R(p \wedge q) = R(p \vee q)$, so the precise form of ψ enters only slightly into the picture.

We make this formal now.

4.3.3.2 Formal definition and results

Let $J \subseteq L$, $X \subseteq M$.

Definition 4.3.2

The supremum (the *lower* bound in interpolation):

(1) The 2-valued case:

(1.1) $X \uparrow J := \{m \in M\colon \exists m'(m \restriction J = m' \restriction J \wedge m' \in X)\}$

(1.2) $M(\phi!\psi) := M(\phi) \uparrow (L - R(\psi))$

(It is another question, whether $M(\phi!\psi)$ is definable - unless, of course, we give the new operator full right as an operator in the language, then $M(\phi!\psi)$ is, by definition, definable. This will also hold in the other cases.)

Remark: ϕ is a finite formula, so $R(\phi)$ is finite, so whether we fix $M(\phi)$ on a finite or infinite set J does not matter. Thus, we could just as well have defined $M(\phi!\psi) = M(\phi) \uparrow R(\psi)$. This is a matter of taste.

(2) The many-valued case:

Given f, define

(2.1) $(f \uparrow J)(m) := sup\{f(m') : m' \in M,\, m \restriction J = m' \restriction J\}$

(2.2) $f_{\phi!\psi} := f_\phi \uparrow (L - R(\psi))$

Definition 4.3.3

The infimum (the *upper* bound in interpolation):

(1) The 2-valued case:

(1.1) $X \downarrow J := \{m \in M\colon \forall m'(m \restriction J = m' \restriction J \Rightarrow m' \in X)\}$

(1.2) $M(\phi?\psi) := M(\phi) \downarrow (L - R(\psi))$

(2) The many-valued case:

Given f, define

(2.1) $(f \downarrow J)(m) := inf\{f(m') : m' \in M, m \upharpoonright J = m' \upharpoonright J\}$

(2.2) $f_{\phi?\psi} := f_\phi \downarrow (L - R(\psi))$

Remark 4.3.1

(1) $\phi!\psi$ and $\phi?\psi$ will give immediately syntactic interpolation, when they are defined (or equivalent to another formula).

(2) The existence (or equivalence) of $\phi!\psi$ and $\phi?\psi$ can be seen as as well-behaviour of a language and logic, as simplified model sets, obtained by simple algebraic operations, are again definable.

□

4.3.3.3 The special case of classical logic.

Fact 4.3.2

(1) $(X \cup X') \uparrow J = (X \uparrow J) \cup (X' \uparrow J)$

(2) $(X \cap X') \downarrow J = (X \downarrow J) \cap (X' \downarrow J)$

(3) Let ϕ and ψ be consistent, but no tautologies, and let ϕ and ψ have no common variables. Let J contain all variables in ϕ, but no variable in ψ.

Then $M(\phi \wedge \psi) \uparrow J = M(\phi)$, and $M(\phi \vee \psi) \downarrow J = M(\phi)$.

Proof

(1) is shown in Fact 4.3.4 (page 140), (8), (2) is shown in Fact 4.3.6 (page 143), (9). As the arguments are simple, we give them here already, for the reader's convenience: (1): Let $m \in (X \cup X') \uparrow J$, so there is $m' \in X \cup X'$ with $m \upharpoonright J = m' \upharpoonright J$, so $m \in (X \uparrow J) \cup (X' \uparrow J)$. The converse is even easier. (2): Let $m \in (X \cap X') \downarrow J$, so for all m' such that $m' \upharpoonright J = m \upharpoonright m$ $m' \in X \cap X'$, so $m \in (X \downarrow J) \cap (X' \downarrow J)$. Again, the converse is even easier.

We turn to (3).

We first show $M(\phi \wedge \psi) \uparrow J = M(\phi)$. $m \in M(\phi \wedge \psi) \uparrow J$ iff there is m', $m \upharpoonright J = m' \upharpoonright J$, $m' \models \phi \wedge \psi$. Suppose $m \in M(\phi \wedge \psi) \uparrow J$, but $m \not\models \phi$. Then any m' such that $m \upharpoonright J = m' \upharpoonright J$ will also make ϕ false, contradiction. Conversely, let $m \models \phi$. Then we can modify m outside J to make ψ true, so $m \in M(\phi \wedge \psi) \uparrow J$.

We now show $M(\phi \vee \psi) \downarrow J = M(\phi)$. $m \in M(\phi \vee \psi) \downarrow J$ iff for all m' such that $m' \upharpoonright J = m \upharpoonright J$, $m' \models \phi \vee \psi$. Suppose $m \in M(\phi \vee \psi) \downarrow J$, but $m \not\models \phi$. Change now m outside J to make ψ false, this is possible, contradiction. Conversely, let $m \models \phi$. Then, no matter how we change m outside J, m' will still be a model of ϕ, and thus of $\phi \vee \psi$.

□

We use the disjunctive and conjunctive normal forms in classical logic to obtain the following result:

Fact 4.3.3

Let the logic be classical propositional logic.

Let $J := \{p_{1,1}, \ldots, p_{1,m_1}, \ldots, p_{n,1}, \ldots, p_{n,m_n}\}$

(1) Let $\phi_i := \pm p_{i,1} \wedge \ldots \wedge \pm p_{i,m_i}$ and $\psi_i := \pm q_{i,1} \wedge \ldots \wedge \pm q_{i,k_i}$, let $\phi := (\phi_1 \wedge \psi_1) \vee \ldots \vee (\phi_n \wedge \psi_n)$. Then $\phi \uparrow J = \phi_1 \vee \ldots \vee \phi_n$.

(2) Let $\phi_i := \pm p_{i,1} \vee \ldots \vee \pm p_{i,m_i}$ and $\psi_i := \pm q_{i,1} \vee \ldots \vee \pm q_{i,k_i}$, let $\phi := (\phi_1 \vee \psi_1) \wedge \ldots \wedge (\phi_n \vee \psi_n)$. Then $\phi \downarrow J = \phi_1 \wedge \ldots \wedge \phi_n$.

Proof

(1) By Fact 4.3.2 (page 139) (1) $M(\phi) \uparrow J = M(\phi_1 \wedge \psi_1) \uparrow J \cup \ldots \cup M(\phi_n \wedge \psi_n) \uparrow J$. By Fact 4.3.2 (page 139) (3), $M(\phi_i \wedge \psi_i) \uparrow J = M(\phi_i)$.

(2) By Fact 4.3.2 (page 139) (2) $M(\phi) \downarrow J = M(\phi_1 \vee \psi_1) \downarrow J \cap \ldots \cap M(\phi_n \vee \psi_n) \downarrow J$. By Fact 4.3.2 (page 139) (3), $M(\phi_i \vee \psi_i) \downarrow J = M(\phi_i)$.

□

4.3.3.4 General results on the new operators

We turn to some general results on the new operators. Note that we work with some redundancy, as the positive cases for the many-valued situation imply the 2-valued result. Still, as the operators are somewhat unusual, we prefer to proceed slowly, and first treat the two-valued case.

Fact 4.3.4

Consider the 2-valued case.

(1) $X \subseteq X \uparrow J \subseteq M$.

(2) $X \uparrow L = X$.

(3) $X \uparrow \emptyset = M$ iff $X \neq \emptyset$.

(4) $\emptyset \uparrow J = \emptyset$.

(5) $M \uparrow J = M$.

(6) $X \subseteq X' \Rightarrow X \uparrow J \subseteq X' \uparrow J$.

(7) $J \subseteq J' \Rightarrow X \uparrow J' \subseteq X \uparrow J$.

(8) $(X \cup X') \uparrow J = (X \uparrow J) \cup (X' \uparrow J)$.

(9) $(X \cap X') \uparrow J \subseteq (X \uparrow J) \cap (X' \uparrow J)$. The converse (i.e., $\supseteq$) is not always true.

(10) $\boldsymbol{C}(X \uparrow J) \subseteq (\boldsymbol{C}X) \uparrow J$. The converse is not always true.

(11) $X \uparrow (J \cup J') \subseteq (X \uparrow J) \cap (X \uparrow J')$. The converse is not always true.

(12) $(X \uparrow J) \cup (X \uparrow J') \subseteq X \uparrow (J \cap J')$. The converse is not always true.

(13) In general, $\boldsymbol{C}(X \uparrow J) \not\subseteq X \uparrow (\boldsymbol{C}J)$. In general, $X \uparrow (\boldsymbol{C}J) \not\subseteq \boldsymbol{C}(X \uparrow J)$.

(14) $(X \uparrow J) \uparrow J' = X \uparrow (J \cap J')$.

(15) $(\phi!p)!q = \phi!p \wedge q$.

(16) $M(\phi!TRUE) = M(\phi)$.

(17) Let γ_X be the characteristic function for X. Then $\gamma_{X\uparrow J}(m) = sup\{\gamma_X(m') : m' \in M, m \restriction J = m' \restriction J\}$.

Proof

(1) – (4) and (6) – (8) are trivial.

(5) follows from (1).

(9) follows from (6).

For the converse: Let $L := \{p, q\}$, $J := \{p\}$. Let $m := p \wedge q$, $m' := p \wedge \neg q$, $X := \{m\}$, $X' := \{m'\}$, so $X \cap X' = \emptyset$, $m \restriction J = m' \restriction J$, but $m, m' \in (X \uparrow J) \cap (X' \uparrow J)$.

(10) Let $m \in M$, $m \notin X \uparrow J \Rightarrow \forall m' \in X.m \restriction J \neq m' \restriction J \Rightarrow m \notin X \Rightarrow m \in \boldsymbol{C}X \subseteq ((\boldsymbol{C}X) \uparrow J)$.

For the converse: Let $L := \{p, q\}$, $J := \{p\}$. Let $X = \{p \wedge q\}$. So $X \uparrow J = \{p \wedge q, p \wedge \neg q\}$, $\boldsymbol{C}(X \uparrow J) = \{\neg p \wedge q, \neg p \wedge \neg q\}$, $\boldsymbol{C}(X) = \{p \wedge \neg q, \neg p \wedge q, \neg p \wedge \neg q\}$, $(\boldsymbol{C}X) \uparrow J = M$.

(11) follows from (7).

For the converse: Let $L := \{p, q\}$, $J := \{p\}$, $J' := \{q\}$. Let $X := \{p \wedge q, \neg p \wedge \neg q\}$, so $p \wedge \neg q \in (X \uparrow J) \cap (X \uparrow J')$, but $p \wedge \neg q \notin X = X \uparrow (J \cup J')$.

(12) follows from (7).

For the converse: Let $L := \{p, q\}$, $J := \{p\}$, $J' := \{q\}$. Let $X := \{p \wedge q\}$. $X \uparrow \emptyset = M$ by (3), but $\neg p \wedge \neg q \notin (X \uparrow J) \cup (X \uparrow J')$.

(13) Consider $L := \{p, q\}$, $J := \{p\}$, $X := \{p \wedge q\}$. Then $X \uparrow J = \{p \wedge q, p \wedge \neg q\}$, so $\mathcal{C}(X \uparrow J) = \{\neg p \wedge q, \neg p \wedge \neg q\}$. $X \uparrow \mathcal{C}J = X \uparrow \{q\} = \{p \wedge q, \neg p \wedge q\}$.

(14)

"$\supseteq$": Let $m \in X \uparrow (J \cap J')$, so there is $m' \in X$, $m \restriction J \cap J' = m' \restriction J \cap J'$. Define m'' on J like m', on $L - J$ like m. (Note that we combine here arbitrarily!) So, as $m' \in X$, $m'' \in X \uparrow J$. We want to show $m \in (X \uparrow J) \uparrow J'$, it suffices to show $m'' \restriction J' = m \restriction J'$. But m'' is on $L - J$ like m, on $J \cap J'$ like m', where m' is like m, so $m'' \restriction J' = m \restriction J'$.

"$\subseteq$": Let $m \in (X \uparrow J) \uparrow J'$, we have to find $m' \in X.m \restriction (J \cap J') = m' \restriction (J \cap J')$. As $m \in (X \uparrow J) \uparrow J'$, there is $m_0 \in X \uparrow J$, $m_0 \restriction J' = m \restriction J'$. As $m_0 \in X \uparrow J$, there is $m_1 \in X$, $m_1 \restriction J = m_0 \restriction J$. So $m_1 \restriction (J \cap J') = m_0 \restriction (J \cap J') = m \restriction (J \cap J')$.

(15) Let $X := M(\phi)$. $(X!p)!q = (X \uparrow (L - \{p\})) \uparrow (L - \{q\}) =$ (by (14)) $X \uparrow (L - \{p, q\}) = \phi!p \wedge q$.

(16) $M(\phi!TRUE) = M(\phi) \uparrow L = M(\phi)$.

(17) $\gamma_{X\uparrow J}(m) = 1$ iff $m \in X \uparrow J$ iff $\exists m' \in X.m \restriction J = m' \restriction J$ iff $\exists m'(\gamma_X(m') = 1$ and $m \restriction J = m' \restriction J)$ iff $sup\{\gamma_X(m') : m' \in M, m \restriction J = m' \restriction J\} = 1$.

□

Fact 4.3.5

Consider the many-valued case.

Let $0 : M \to V$ be the constant 0 (= minimal value) function, $1 : M \to V$ be the constant 1 (= maximal value) function.

The same counterexamples as for the 2-valued case work, we just take the characteristic functions.

(1) $f \leq f \uparrow J \leq 1$.

(2) $f \uparrow L = f$.

(4) $0 \uparrow J = 0$.

(5) $1 \uparrow J = 1$.

(6) $f \leq g \Rightarrow f \uparrow J \leq g \uparrow J$.

(7) $J \subseteq J' \Rightarrow f \uparrow J' \leq f \uparrow J$.

(8) $(sup(f,g)) \uparrow J = sup(f \uparrow J, g \uparrow J)$.

(9) $(inf(f,g)) \uparrow J \leq inf(f \uparrow J, g \uparrow J)$. The converse is not always true.

(11) $f \uparrow (J \cup J') \leq inf(f \uparrow J, f \uparrow J')$. The converse is not always true.

(12) $sup(f \uparrow J, f \uparrow J') \leq f \uparrow (J \cap J')$. The converse is not always true.

(14) $(f \uparrow J) \uparrow J' = f \uparrow (J \cap J')$.

(15) $(\phi!p)!q = \phi!p \wedge q$.

(16) $f_{\phi!TRUE} = f_\phi$.

Proof

(1) − (2), (4), (6) − (8) are trivial.

(5) follows from (1).

(9) follows from (6).

(11) by (7).

(12) by (7).

(14)

For simplicity, we write just $f \uparrow J \cap J'(m)$ for $(f \uparrow J \cap J')(m)$, $(f \uparrow J) \uparrow J'(m)$ for $((f \uparrow J) \uparrow J')(m)$, etc.

"$\geq$": Consider $f \uparrow J \cap J'(m)$. We have to show $(f \uparrow J) \uparrow J'(m) \geq f \uparrow (J \cap J')(m)$. Let m' be such that $m' \upharpoonright J \cap J' = m \upharpoonright J \cap J'$, $f \uparrow J \cap J'(m) = f(m')$ (i.e., m' has maximal value among such m'). (Use again arbitrary combinations.) Define m'' on J like m', on $L - J$ like m, so, as m'' is like m' on J, $f \uparrow J(m'') := max\{f(m_0) : m_0 \upharpoonright J = m'' \upharpoonright J\} \geq f(m')$. Note that m'' is on $L - J$ like m, on $J \cap J'$ like m', where m' is like m, so $m'' \upharpoonright J' = m \upharpoonright J'$, so $(f \uparrow J) \uparrow J'(m) := max\{f \uparrow J(m_0) : m_0 \upharpoonright J' = m \upharpoonright J'\} \geq f \uparrow J(m'')$. Thus, $(f \uparrow J) \uparrow J'(m) \geq f \uparrow J(m'') \geq f(m') = f \uparrow J \cap J'(m)$.

"$\leq$": Consider $(f \uparrow J) \uparrow J'(m)$. Let m' be such that $m \upharpoonright J' = m' \upharpoonright J'$, $(f \uparrow J) \uparrow J'(m) = (f \uparrow J)(m')$, i.e., m' has maximal value. Let m'' be such that $m'' \upharpoonright J = m' \upharpoonright J$, and $(f \uparrow J)(m') = f(m'')$, i.e., m'' has maximal value. Then $m'' \upharpoonright J \cap J' = m' \upharpoonright J \cap J' = m \upharpoonright J \cap J'$, so $f \uparrow J \cap J'(m)$

$\geq f(m'') = f \uparrow J(m') = (f \uparrow J) \uparrow J'(m)$.

(15) Let $f := f_\phi$. $(f!p)!q = (f \uparrow (L - \{p\})) \uparrow (L - \{q\}) =$ (by (14)) $f \uparrow (L - \{p, q\}) = f!p \wedge q$.

(16) $f_{\phi!TRUE} = f_\phi \uparrow L = f_\phi$.

□

Fact 4.3.6

Consider the 2-valued case.

(1) $X \downarrow J \subseteq X$.

(2) $X \downarrow L = X$.

(3) $X \downarrow \emptyset = \emptyset$ iff $X \neq M$, and M iff $X = M$.

(4) $\emptyset \downarrow J = \emptyset$.

(5) $M \downarrow J = M$.

(6) $X \subseteq X' \Rightarrow X \downarrow J \subseteq X' \downarrow J$.

(7) $J \subseteq J' \Rightarrow X \downarrow J \subseteq X \downarrow J'$.

(8) $(X \downarrow J) \cup (X' \downarrow J) \subseteq (X \cup X') \downarrow J$. The converse is not always true.

(9) $(X \cap X') \downarrow J = (X \downarrow J) \cap (X' \downarrow J)$.

(10) $(\boldsymbol{C}X) \downarrow J \subseteq \boldsymbol{C}(X \downarrow J)$. The converse is not always true.

(11) $(X \downarrow J) \cup (X \downarrow J') \subseteq X \downarrow (J \cup J')$. The converse is not always true.

(12) $X \downarrow (J \cap J) \subseteq (X \downarrow J) \cap (X \downarrow J')$. The converse is not always true.

(13) In general, $\boldsymbol{C}(X \downarrow J) \not\subseteq X \downarrow (\boldsymbol{C}J)$ and $X \downarrow (\boldsymbol{C}J) \not\subseteq \boldsymbol{C}(X \downarrow J)$.

(14) $(X \downarrow J) \downarrow J' = X \downarrow (J \cap J')$

(15) $(\phi?p)?q = \phi?p \wedge q$.

(16) $M(\phi?TRUE) = M(\phi)$.

(17) Let γ_X be the characteristic function for X. Then $\gamma_{X \downarrow J}(m) = inf\{\gamma_X(m') : m' \in M, m \upharpoonright J = m' \upharpoonright J\}$.

Proof

(1) – (6) are trivial.

(7) $m \in X \downarrow J \Rightarrow \forall m'(m \upharpoonright J = m' \upharpoonright J \Rightarrow m' \in X)$. But $m \upharpoonright J' = m' \upharpoonright J' \Rightarrow m \upharpoonright J = m' \upharpoonright J \Rightarrow m' \in X$, so $m \in X \downarrow J'$.

(8) by (6).

For the converse: Let $L := \{p, q\}$, $J := \{p\}$. Let $X = \{p \wedge q\}$. Let $X' := \{p \wedge \neg q\}$. So $X \downarrow J = X' \downarrow J = \emptyset$, $(X \cup X') \downarrow J = \{p \wedge q, p \wedge \neg q\}$.

(9)

"$\subseteq$" by (6).

"$\supseteq$" Let $m \in (X \downarrow J) \cap (X' \downarrow J)$, so all m' such that $m' \upharpoonright J = m \upharpoonright J$ are in X, and in X', so they are in $X \cap X'$, so $m \in (X \cap X') \downarrow J$.

(10) $m \in (\boldsymbol{C}X) \downarrow J \Rightarrow m \notin X \Rightarrow \exists m'(m \upharpoonright J = m' \upharpoonright J \wedge m' \notin X)$, so $m \notin X \downarrow J$.

For the converse: Let $L := \{p, q\}$, $J := \{p\}$. Let $X = \{p \wedge q\}$. $X \downarrow J = \emptyset$, so $\boldsymbol{C}(X \downarrow J) = M$. $\boldsymbol{C}(X)$ $= \{p \wedge \neg q, \neg p \wedge q, \neg p \wedge \neg q\}$, $(\boldsymbol{C}X) \downarrow J = \{\neg p \wedge q, \neg p \wedge \neg q\}$.

(11) by (7).

For the converse: Let $L := \{p, q\}$, $J := \{p\}$, $J' := \{q\}$. Let $X = \{p \wedge q\}$. $X \downarrow (J \cup J') = X$, $X \downarrow J = X \downarrow J' = \emptyset$.

(12) by (7).

For the converse: Let $L := \{p, q\}$, $J := \{p\}$, $J' := \{q\}$. Let $X = \{p \wedge q, p \wedge \neg q, \neg p \wedge q\}$. $X \downarrow \emptyset = \emptyset$ by (3). $X \downarrow J = \{p \wedge q, p \wedge \neg q\}$, $X \downarrow J' = \{p \wedge q, \neg p \wedge q\}$, $(X \downarrow J) \cap (X \downarrow J') = \{p \wedge q\}$.

(13)

"$\not\subseteq$": Let $J := \emptyset$, $X \neq M$, so $\boldsymbol{C}J = L$, $X \downarrow \boldsymbol{C}J = X$, as $X \downarrow \emptyset = \emptyset$, $\boldsymbol{C}(X \downarrow J) = M$.

"$\not\supseteq$": Let $J := \emptyset$. $M \downarrow (\boldsymbol{C}J) = M$, $M \downarrow J = M$, so $\boldsymbol{C}(M \downarrow J) = \emptyset$.

(14)

"$\subseteq$": Let $m \in (X \downarrow J) \downarrow J'$, we have to show $m \in X \downarrow (J \cap J')$, i.e., for all m'' such that $m'' \upharpoonright J \cap J' = m \upharpoonright J \cap J'$ $m'' \in X$. Take m'' such that $m \upharpoonright (J \cap J') = m'' \upharpoonright (J \cap J')$. Define m' such that m' on J' is like m, m' on $L - J'$ is like m''. Then by $m \in (X \downarrow J) \downarrow J'$ and $m \upharpoonright J' = m' \upharpoonright J'$, $m' \in X \downarrow J$. Note that m'' on J is like m' : On $L - J'$ m' is like m''. On $J \cap J'$, m' is like m, and m is like m'', so m'' is like m'. As $m' \in X \downarrow J$, $m'' \upharpoonright J = m' \upharpoonright J$, $m'' \in X$ by definition.

"$\supseteq$": Let $m \in X \downarrow (J \cap J')$. So, if $m' \upharpoonright (J \cap J') = m \upharpoonright (J \cap J')$, $m' \in X$. We have to show $m'' \upharpoonright J' = m \upharpoonright J' \Rightarrow m'' \in X \downarrow J$, i.e. if $m'' \upharpoonright J' = m \upharpoonright J'$ and $m_0 \upharpoonright J = m'' \upharpoonright J$, then $m_0 \in X$. But $m_0 \upharpoonright (J \cap J') = m'' \upharpoonright (J \cap J') = m \upharpoonright (J \cap J')$, so $m_0 \in X$ by prerequisite.

(15) Let $X := M(\phi)$. $(\phi?p)?q = (X \downarrow (L - \{p\})) \downarrow (L - \{q\}) =$ (by (14)) $X \downarrow (L - \{p, q\}) = \phi \downarrow p \wedge q$.

(16) $M(\phi?TRUE) = M(\phi) \downarrow L = M(\phi)$.

(17) $\gamma_{X \downarrow J}(m) = 1$ iff $m \in X \downarrow J$ iff $\forall m' \in M(m \upharpoonright J = m' \upharpoonright J \Rightarrow m' \in X)$ iff $inf\{\gamma_X(m') : m' \in M$, $m \upharpoonright J = m' \upharpoonright J\} = 1$.

□

Fact 4.3.7

Consider the many-valued case.

Let $0 : M \to V$ be the constant 0 (= minimal value) function, $1 : M \to V$ be the constant 1 (= maximal value) function.

The same counterexamples as for the 2-valued case work, we just take the characteristic functions.

(1) $f \downarrow J \leq f$.

(2) $f \downarrow L = f$.

(4) $0 \downarrow J = 0$.

(5) $1 \downarrow J = 1$.

(6) $f \leq g \Rightarrow f \downarrow J \leq g \downarrow J$.

(7) $J \subseteq J' \Rightarrow f \downarrow J \leq f \downarrow J'$.

(8) $sup(f \downarrow J, g \downarrow J) \leq sup(f,g) \downarrow J$, the converse is not always true.

(9) $inf(f,g) \downarrow J = inf(f \downarrow J, g \downarrow J)$.

(11) $sup(f \downarrow J, f \downarrow J') \leq f \downarrow (J \cup J')$, the converse is not always true.

(12) $f \downarrow (J \cap J) \leq inf(f \downarrow J, f \downarrow J')$, the converse is not always true.

(14) $(f \downarrow J) \downarrow J' = f \downarrow (J \cap J')$

(15) $(\phi?p)?q = \phi?p \wedge q$

(16) $f_{\phi?TRUE} = f_\phi$.

Proof

(1) – (6) are trivial.

(7) $m \in X \downarrow J \Rightarrow \forall m'(m \upharpoonright J = m' \upharpoonright J \Rightarrow m' \in X)$. But $m \upharpoonright J' = m' \upharpoonright J' \Rightarrow m \upharpoonright J = m' \upharpoonright J \Rightarrow m' \in X$, so $m \in X \downarrow J'$.

(8) by (6).

(9)

"$\leq$" by (6).

"$\geq$": Suppose $(inf(f,g) \downarrow J)(m) < inf(f \downarrow J(m), g \downarrow J(m))$. Choose m' such that $m \upharpoonright J = m' \upharpoonright J$ and $inf(f,g)(m') = inf(f,g) \downarrow J(m)$. Note that $f \downarrow J(m) \leq f(m')$, $g \downarrow J(m) \leq g(m')$. Suppose without loss of generality $f(m') \leq g(m')$. Then $f(m') = inf(f,g)(m') = (inf(f,g) \downarrow J)(m) < inf(f \downarrow J(m), g \downarrow J(m)) \leq f \downarrow J(m)$, a contradiction.

(11) by (7).

(12) by (7).

(14)

"$\leq$": We have to show $(f \downarrow J) \downarrow J'(m) \leq f \downarrow (J \cap J')(m)$. Let m'' be such that $m \upharpoonright J \cap J' = m'' \upharpoonright J \cap J'$, we have to show $(f \downarrow J) \downarrow J'(m) \leq f(m'')$, then $(f \downarrow J) \downarrow J'(m) \leq f \downarrow J \cap J'(m) := min\{f(m'') : m \upharpoonright J \cap J' = m'' \upharpoonright J \cap J'\}$. Take such m''. Define m' such that m' on J' is like m, m' on $L - J'$ is like m''. As $m' \upharpoonright J' = m \upharpoonright J'$, $(f \downarrow J) \downarrow J'(m) \leq f \downarrow J(m')$. Note that m'' is like m' on J : On $L - J'$, m' is like m''. On $J \cap J'$, m' is like m, and m is like m'', so m'' is like m'. Thus $f \downarrow J(m') \leq f(m'')$, so $(f \downarrow J) \downarrow J'(m) \leq f \downarrow J(m') \leq f(m'')$.

"$\geq$": We show $f \downarrow J \cap J'(m) \leq (f \downarrow J) \downarrow J'(m)$. We have to show that $min\{f \downarrow J(m') : m' \upharpoonright J' = m \upharpoonright J'\} \geq f \downarrow J \cap J'(m)$. So take any m' such that $m' \upharpoonright J' = m \upharpoonright J'$. We have to show that $f \downarrow J(m') \geq f \downarrow J \cap J'(m)$, i.e., for all m'' such that $m'' \upharpoonright J = m' \upharpoonright J$, $f(m'') \geq f \downarrow J \cap J'(m) := min\{f(m_0) : m_0 \upharpoonright J \cap J' = m \upharpoonright J \cap J'\}$. If $m'' \upharpoonright J = m' \upharpoonright J$, then $m'' \upharpoonright J \cap J' = m' \upharpoonright J \cap J' = m \upharpoonright J \cap J'$, so m'' is one of those m_0, and we are done.

(15) Let $f := f_\phi$. (f?p)?q = $(f \downarrow (L - \{p\})) \downarrow (L - \{q\})$ = (by (14)) $f \downarrow (L - \{p,q\}) = f \downarrow p \wedge q$.

(16) $f_{\phi ?TRUE} = f_\phi \downarrow L = f_\phi$.

□

4.4 Monotone and antitone syntactic interpolation

4.4.1 Introduction

We have shown *semantic* interpolation, this is not yet *syntactic* interpolation. We still need that the set of sequences is definable. (The importance of definability in the context of non-monotonic logics was examined by one of the authors in [Sch92].) Note that we "simplified" the set of sequences when going from Σ' to Σ'', in Proposition 4.2.1 (page 131), but perhaps the logic at hand does not share this perspective. Consider, e.g., intuitionistic logic with three worlds, linearly ordered. This is a monotonic logic, so by our results, it has semantic interpolation. But it has no syntactic interpolation, so the created set of models must not be definable, see Example 4.4.1 (page 157). In classical propositional logic, the created set *is* definable, as we will see in Proposition 4.4.1 (page 147).

But first a side remark.

Comment 4.4.1

Usual approaches to repair interpolation construct a chain of ever richer languages for interpolation:

We can go further with a logic in language L0 for which there is no interpolation. For every pair of formulas which give a counterexample to interpolation we introduce a new connective which corresponds to the semantic interpolant. Now we have a language L1 which allows for interpolation for formulas in the original language L0. L1 itself may or may not have interpolation. So we might have to continue to L2, L3, etc.

This leads to the following definition for the cases where we have semantic, but not necessarily syntactic interpolation. We have, however, not examined this notion.

Definition 4.4.1

A logic has level 0 semantic interpolation, iff it has interpolation.

A logic has level $n+1$ semantic interpolation iff it has no level n semantic interpolation, but introducing new elements into the language (of level n) results in interpolation also for the new language.

The case of full non-monotonic logic is, of course, different, as the logics might not even have semantic interpolation, so above repairing might not be possible.

4.4.2 The classical propositional case

We saw already in Section 4.3.3 (page 137) the general result for classical logic, we give here a more special argument. We show that semantic interpolation carries over to syntactic interpolation in classical propositional logic. What we see as simplification of sets of sequences, is seen the same way by classical logic. If the first is definable, so is the second.

Nicht schon irgendwo???

Proposition 4.4.1

Simplification (i.e. sup) preserves definability in classical propositional logic:

Let $\Gamma = \Sigma \restriction X' \times \Pi X''$. Then, if Σ is formula definable, so is Γ.

Proof

As Σ is formula definable, it is defined by $\phi_1 \vee \ldots \vee \phi_n$, where $\phi_i = \psi_{i,1} \wedge \ldots \wedge \psi_{i,n_i}$. Let $\Phi_i := \{\psi_{i,1}, \ldots, \psi_{i,n_i}\}$, $\Phi'_i := \{\psi \in \Phi_i : \psi \in X'\}$ (more precisely, ψ or $\neg\psi \in X'$), $\Phi''_i := \Phi_i - \Phi'_i$. Let $\phi'_i := \bigwedge \Phi'_i$. Thus $\phi_i \vdash \phi'_i$. We show that $\phi'_1 \vee \ldots \vee \phi'_n$ defines Γ. (Alternatively, we may replace all $\psi \in \Phi''_i$ by TRUE.)

(1) $\Gamma \models \phi'_1 \vee \ldots \vee \phi'_n$

Lt $\sigma \in \Gamma$, then there is $\tau \in \Sigma$ s.t. $\sigma \restriction X' = \phi \restriction X'$. By prerequisite, $\tau \models \phi_1 \vee \ldots \vee \phi_n$, so $\tau \models \phi'_1 \vee \ldots \vee \phi'_n$, so $\sigma \models \phi'_1 \vee \ldots \vee \phi'_n$.

(2) Suppose $\sigma \notin \Gamma$, we have to show $\sigma \not\models \phi'_1 \vee \ldots \vee \phi'_n$.

Suppose then $\sigma \notin \Gamma$, but $\sigma \models \phi'_1 \vee \ldots \vee \phi'_n$, without loss of generality $\sigma \models \phi'_1 = \bigwedge \Phi'_1$. As $\sigma \notin \Gamma$, there is no $\tau \in \Sigma$ $\tau \restriction X' = \sigma \restriction X'$. Choose τ s.t. $\sigma \restriction X' = \tau \restriction X'$ and $\tau \models \psi$ for all $\psi \in \Phi''_1$. By $\sigma \models \psi$ for $\psi \in \Phi'_1$, and $\sigma \restriction X' = \tau \restriction X'$ $\tau \models \psi$ for $\psi \in \Phi'_1$. By prerequisite, $\tau \models \psi$ for $\psi \in \Phi''_1$, so $\tau \models \psi$ for all $\psi \in \Phi_1$, so $\tau \models \phi_1 \vee \ldots \vee \phi_n$, and $\tau \in \Sigma$, as $\phi \vee \ldots \vee \phi$ defines Σ, contradiction.

□

Corollary 4.4.2

The same result holds if Σ is theory definable.

Proof

(Outline). Define Σ by a (possibly infinite) conjunction of (finite) disjunctions. Transform this into a possibly infinite disjunction of possibly infinite conjunctions. Replace all $\phi \in \Phi''_i$ by TRUE. The same proof as above shows that this defines Γ (finiteness was nowhere needed). Transform backward into a conjunction of finite disjunctions, where the $\phi \in \Phi''_i$ are replaced by TRUE.

□

We could have used the following trivial fact for the proof of Proposition 4.4.1 (page 147):

Fact 4.4.3

Let $\Sigma = \Sigma_1 \cup \Sigma_2$. Then $\Sigma \restriction X' \times \Pi X'' = (\Sigma_1 \restriction X' \times \Pi X'') \cup (\Sigma_2 \restriction X' \times \Pi X'')$.

Proof

"$\subseteq$": Let $\sigma \in \Sigma \restriction X' \times \Pi X''$, then there is $\sigma' \in \Sigma$ s.t. $\sigma \restriction X' = \sigma' \restriction X'$, so $\sigma' \in \Sigma_1$ or $\sigma' \in \Sigma_2$. If $\sigma' \in \Sigma_1$, then $\sigma \in \Sigma_1 \restriction X' \times \Pi X''$, likewise for $\sigma' \in \Sigma_2$.

The converse is even more trivial.

□

Remark 4.4.4

An analogous result about interesection does not hold, of course: $(\Sigma_1 \restriction X' \times X'') \cap (\Sigma_2 \restriction X' \times X'')$ might well be $\neq \emptyset$, but $\Sigma_1 \cap \Sigma_2 = \emptyset$.

4.4.3 General finite (intuionistic) Goedel logics

The semantics is a linearly ordered finite set of worlds, with increasing truth, as usual in intuitionistic logics. Let n be the number of worlds, then ϕ has truth value k in the structure iff ϕ holds from world $n+1-k$ onward (and 0 iff it never holds). Thus, if it holds everywhere, it has truth value n, if it holds from world 2 onward, it has value $n-1$, etc. It is well known that such logics have (syntactic) interpolation for $n=2$, but not for $n>2$. This is the reason we treat them here. We will connect the interpolation problem to the existence of normal forms. The connection is incomplete, as we will show that suitable normal forms entail interpolation, but we do not know if this condition is necessary.

Definition 4.4.2

Finite intuitionistic Goedel logics with $n+1$ truth values $FALSE = 0 \leq 1 \leq \ldots \leq n = TRUE$ are defined as follows:

(1) $f_{\phi \wedge \psi}(m) := inf\{f_\phi(m), f_\psi(m)\}$,

(2) $f_{\phi \vee \psi}(m) := sup\{f_\phi(m), f_\psi(m)\}$,

(3) negation $\neg$ is defined by:

$$f_{\neg\phi}(m) := \begin{cases} TRUE & iff \quad f_\phi(m) = FALSE \\ FALSE & \quad otherwise \end{cases}$$

(4) implication $\rightarrow$ is defined by:

$$f_{\phi \rightarrow \psi}(m) := \begin{cases} TRUE & iff \quad f_\phi(m) \leq f_\psi(m) \\ f_\psi(m) & \quad otherwise \end{cases}$$

Thus, for $n+1=2$, this is classical logic. So we assume now $n \geq 2$.

Definition 4.4.3

We will also consider the following additional operators:

(1) J is defined by:

$$f_{J\phi}(m) := \begin{cases} f_\phi(m) & iff \quad f_\phi(m) = FALSE \quad or \quad f_\phi(m) = TRUE \\ f_\phi(m) + 1 & \quad otherwise \end{cases}$$

The intuitive meaning is: “it holds in the next moment”

(2) A is defined by:

$$f_{A(\phi)}(m) := \begin{cases} TRUE & iff \quad f_\phi(m) = TRUE \\ FALSE & \quad otherwise \end{cases}$$

Thus, A is the dual of negation, we might call it affirmation.

(3) F is defined by:

$$f_{F\phi}(m) := \begin{cases} FALSE & iff \quad f_\phi(m) = FALSE \quad or \quad f_\phi(m) = TRUE \\ f_\phi(m) + 1 & \quad otherwise \end{cases}$$

The intuitive meaning is: “it begins to hold in the next moment”

(4) Z (cyclic addition of 1) is defined by:

$$f_{Z\phi}(m) := \begin{cases} FALSE & iff \quad f_\phi(m) = TRUE \\ f_\phi(m) + 1 & \quad otherwise \end{cases}$$

Note that Z is slightly different from J. We do not know if there is an intuitive meaning.

To help the intuition, we give the truth tables of the basic operators for $n = 3$, and of $\neg, \rightarrow, J, A, F, Z$ for $n = 4$ and $n = 6$.

		b		0	1	2		0	1	2		0	1	2		0	1	2
a	$\neg a$		$a \rightarrow b$				$a \wedge b$				$a \vee b$				$a \leftrightarrow b$			
0	2			2	2	2		0	0	0		0	1	2		2	0	0
1	0			0	2	2		0	1	1		1	1	2		0	2	1
2	0			0	1	2		0	1	2		2	2	2		0	1	2

						b		0	1	2	3
a	$\neg a$	Ja	Aa	Fa	Za		$a \to b$				
0	3	0	0	0	1			3	3	3	3
1	0	2	0	2	2			0	3	3	3
2	0	3	0	3	3			0	1	3	3
3	0	3	3	0	0			0	1	2	3

						b		0	1	2	3	4	5
a	$\neg a$	Ja	Aa	Fa	Za		$a \to b$						
0	5	0	0	0	1			5	5	5	5	5	5
1	0	2	0	2	2			0	5	5	5	5	5
2	0	3	0	3	3			0	1	5	5	5	5
3	0	4	0	4	4			0	1	2	5	5	5
4	0	5	0	5	5			0	1	2	3	5	5
5	0	5	5	0	0			0	1	2	3	4	5

4.4.3.1 The basic operators $\wedge, \vee, \to, \neg$

We work now towards a suitable normal form, even though we cannot obtain it for $n > 3$. This will also indicate a way to repair those logics by introducing suitable additional operators, which allow to obtain such normal forms.

We have the following fact:

Fact 4.4.5

(0) With one variable a we can define up to semantical equivalence exactly the following 6 different formulas:

a, $\neg a$, $\neg\neg a$, $TRUE = a \to a$, $FALSE = \neg(a \to a)$, $\neg\neg a \to a$.

The following semantic equivalences hold:

(Note: all except (14) hold also for 4 and 6 truth values, so probably for arbitrarily many truth values, but this is not checked so far.)

Triple negation can be simplified:

(1) $\neg\neg\neg a \leftrightarrow \neg a$

Disjunction and conjunction combine classically:

(2) $\neg(a \vee b) \leftrightarrow \neg a \wedge \neg b$

(3) $\neg(a \wedge b) \leftrightarrow \neg a \vee \neg b$

(4) $a \wedge (b \vee c) \leftrightarrow (a \wedge b) \vee (a \wedge c)$

(5) $a \vee (b \wedge c) \leftrightarrow (a \vee b) \wedge (a \vee c)$

Implication can be eliminated from combined negation and implication:

(6) $\neg(a \to b) \leftrightarrow \neg\neg a \wedge \neg b$

(7) $(a \to \neg b) \leftrightarrow (\neg a \vee \neg b)$

(8) $(\neg a \to b) \leftrightarrow (\neg\neg a \vee b)$

Implication can be put inside when combined with $\wedge$ and $\vee$:

(9) $(a \vee b \to c) \leftrightarrow ((a \to c) \wedge (b \to c))$

(10) $(a \wedge b \to c) \leftrightarrow ((a \to c) \vee (b \to c))$

(11) $(a \to b \wedge c) \leftrightarrow ((a \to b) \wedge (a \to c))$

(12) $(a \to b \vee c) \leftrightarrow ((a \to b) \vee (a \to c))$

Nested implication can be flattened for nesting on the right:

(13) $(a \to (b \to c)) \leftrightarrow ((a \wedge b \to c) \wedge (a \wedge \neg c \to \neg b))$

Proof

We use T for TRUE, F for FALSE.

(0) The truth table for the 6 formulas is given by the following table:

a	a	$\neg a$	$\neg\neg a$	$T\ (= a \to a)$	$F\ (= \neg(a \to a))$	$\neg\neg a \to a$
0	0	n	0	n	0	n
1	1	0	n	n	0	1
2	2	0	n	n	0	2
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	..
n	n	0	n	n	0	n

We see that the first line takes the values 0 and n, and the $n-1$ other lines take the vectors of values $(1, \ldots n)$, $(0, \ldots, 0)$, $(n, \ldots, n)$, and that all combinations of first line values and those vectors occur. Thus, we can check closure separately for the first line and the other lines, which is now trivial.

(1) Trivial.

(2) + (3) Both sides can only be T or F. (2): Suppose $a \leq b$, then $\neg(a \vee b) = T$ iff $b = F$, and $\neg a \wedge \neg b = T$ iff $b = F$. The case $b \leq a$ is symmetrical. (3): similar: $\neg(a \wedge b) = T$ iff $a = F$ and $\neg a \vee \neg b = T$ iff $a = F$.

(4) Suppose $b \leq c$. Thus $(a \wedge b) \vee (a \wedge c) = a \wedge c$. If $a \leq c$, then $a \wedge (b \vee c) = a$, else $a \wedge (b \vee c) = c$. The case $c \leq b$ is symmetrical.

(5) Suppose $b \leq c$. Then $(a \vee b) \wedge (a \vee c) = a \vee b$. If $a \leq b$, then $a \vee (b \wedge c) = b$, else $a \vee (b \wedge c) = a$.

(6) Both sides are T or F. $\neg(a \to b) = T$ iff $a > b$ and $b = F$. $\neg\neg a \wedge \neg b = T$ iff $b = F$ and $\neg\neg a = F$. $\neg\neg a = F$ iff $a > F$.

(7) Again, both sides are T or F. $a \to \neg b = T$ iff $a \leq \neg b$ iff $b = F$ or $a = F$.

(8) $\neg a \to b = T$ iff $a > F$ or $b = T$. If $a > F$, then $\neg a \to b = T$ and $\neg\neg a \vee b = T$. If $a = F$, then $\neg a \to b = b$, and $\neg\neg a \vee b = b$.

(9) $a \vee b \to c$ is $a \to c$ or $b \to c$. If $a \leq b$, then it is $b \to c$, and $a \to c \geq b \to c$. The case $b \leq a$ is symmetrical.

(10) – (12) are similar to (9), e.g., (11): If $b \leq c$, then $(a \to b) \wedge c = a \to b$, and $(a \to b) \wedge (a \to c) = a \to b$.

(13) Case 1. $b \leq c$: Then $\neg c \leq \neg b$, and $a \to (b \to c) = T$, $a \wedge b \to c = T$, $a \wedge \neg c \to \neg b = T$.

Case 2. $b > c$: So $(a \to (b \to c)) = (a \to c)$, and $\neg b = F$.

Case 2.1, $a \leq b$: So $a \wedge b \to c = a \to c$.

Case 2.1.1. $\neg c = F$: so $a \wedge \neg c \to \neg b = T$, and we are done.

Case 2.1.2. $\neg c = T$: So $c = F$, $a \to c = a \to F$, and $a \wedge \neg c \to \neg b = a \to \neg b = a \to F$, and we are done again.

Case 2.2. $a > b$: So $a > b > c$, $a \to c = c$, $a \wedge b \to c = b \to c = c$, and $\neg b = F$.

Case 2.2.1. $\neg c = F$: So $a \wedge \neg c \to \neg b = T$, and we are done.

Case 2.2.2. $\neg c = T$: So $c = F$. Thus $a \wedge \neg c \to \neg b = a \to \neg b = a \to F$. But also $a \to c = a \to F$, and we are done again.

□

We assume now that

(Assumption) Any formula of the type $(\phi \to \phi') \to \psi$ is equivalent to a formula Φ containing only flat $\to$'s.

We will later show that this is true for $n = 2$.

Fact 4.4.6

Let above assumption be true. Then:

Every formula ϕ can be transformed into a semantically equivalent formula ψ of the following form:

(1) ψ has the form $\phi_1 \vee \ldots \vee \phi_n$

(2) every ϕ_i has the form $\phi_{i,1} \wedge \ldots \wedge \phi_{i,m}$

(3) every $\phi_{i,m}$ has one of the following forms:

p, or $\neg p$, or $\neg\neg p$, or $p \to q$ - where p and q are propositional variables.

Note that also $\phi \to \phi = TRUE$ can be replaced by $\neg a \vee \neg\neg a$.

Proof

The numbers refer to Fact 4.4.5 (page 150).

We first push $\neg$ downward, towards the interior:

- $\neg(\phi \wedge \psi)$ is transformed to $\neg\phi \vee \neg\psi$ by (3).
- $\neg(\phi \vee \psi)$ is transformed to $\neg\phi \wedge \neg\psi$ by (2).
- $\neg(\phi \to \psi)$ is transformed to $\neg\neg\phi \wedge \neg\psi$ by (6).

We next eliminate any $\phi \to \psi$ where ϕ and ψ are not propositional variables:

- $\neg\phi \to \psi$ is transformed to $\neg\neg\phi \vee \psi$ by (8).

- $\phi \wedge \phi' \to \psi$ is transformed to $(\phi \to \psi) \vee (\phi' \to \psi)$ by (10).
- $\phi \vee \phi' \to \psi$ is transformed to $(\phi \to \psi) \wedge (\phi' \to \psi)$ by (9).
- $(\phi \to \phi') \to \psi$ is transformed to flat Φ by the assumption.
- $\phi \to \neg\psi$ is transformed to $\neg\phi \vee \neg\psi$ by (7).
- $\phi \to \psi \wedge \psi'$ is transformed to $(\phi \to \psi) \wedge (\phi \to \psi')$ by (11).
- $\phi \to \psi \vee \psi'$ is transformed to $(\phi \to \psi) \vee (\phi \to \psi')$ by (12).
- $\phi \to (\psi \to \psi')$ is transformed to $(\phi \wedge \psi \to \psi') \wedge (\phi \wedge \neg\psi' \to \neg\psi)$ by (13).

Finally, we push $\wedge$ inside:

$\phi \wedge (\psi \vee \psi')$ is transformed to $(\phi \wedge \psi) \vee (\phi \wedge \psi')$ by (4).

The exact proof is, of course, by induction.

□

This normal form allows us to use the following facts:

Fact 4.4.7

We will now work for syntactic interpolation. For this purpose, we show that, if f is definable in Proposition 4.2.3 (page 134) , i.e. there is ϕ with $f = f_\phi$, then f^+ in the same Proposition is also definable. Recall that $f^+(m)$ was defined as the maximal $f(m')$ for $m' \restriction J' = m \restriction J'$. We use the normal form just shown, to show that conjuncts and disjuncts can be treated separately.

Our aim is to find a formula which characterizes the maximum. More precisely, if $f = f_\phi$ for some ϕ, we look for ϕ' such that $f_{\phi'}(m) = max\{f_\phi(m') : m' \in M, m \restriction J = m' \restriction J\}$.

First, a trivial fact, which shows that we can treat the elements of J (or $L-$J) one after the other: $max\{g(x, y) : x \in X, y \in Y\} = max\{max\{g(x, y) : x \in X\} : y \in Y\}$. (Proof: The interior max on the right hand side range over subsets of $X \times Y$, so they are all $\leq$ than the left hand side. Conversely, the left hand max is assumed for some $\langle x, y\rangle$, which also figures on the right hand side. A full proof would be an induction.)

Next, we show that we can treat disjunctions separately for one $x \in L$, and also conjunctions, as long as x occurs only in one of the conjuncts. Again, a full proof would be by induction, we only show the crucial arguments. First, some notation:

Notation 4.4.1

(1) We write $m =_{(x)} m'$ as shorthand for $m \restriction (L - \{x\}) = m' \restriction (L - \{x\})$

(2) Let $f : M \to V$, $x \in L$, then $f_{(x)}(m) := max\{f(m') : m' \in M, m =_{(x)} m'\}$.

(3) Let $f_\phi : M \to V$, and $(f_\phi)_{(x)} = f_{\phi'}$ for some ϕ', then we write $\phi_{(x)}$ for (some such) ϕ'.

Fact 4.4.8

(1) If $\phi = \phi' \vee \phi''$, and $\phi'_{(x)}$, $\phi''_{(x)}$ both exist, then so does $\phi_{(x)}$, and $\phi_{(x)} = \phi'_{(x)} \vee \phi''_{(x)}$.

(2) If $\phi = \phi' \wedge \phi''$, $\phi'_{(x)}$ exists, and ϕ'' does not contain x, then $\phi_{(x)}$ exists, and $\phi_{(x)} = \phi'_{(x)} \wedge \phi''$.

Proof

(1) We have to show $f_{\phi_{(x)}} = f_{(\phi'_{(x)} \vee \phi''_{(x)})}$.

By definition of validity of $\vee$, we have $f_{(\phi'_{(x)} \vee \phi''_{(x)})}(m) = max\{f_{\phi'_{(x)}}(m), f_{\phi''_{(x)}}(m)\}$. $f_{\phi_{(x)}}(m) := max\{f_\phi(m') : m' =_{(x)} m\}$, so $f_{(\phi'_{(x)} \vee \phi''_{(x)})}(m) = max\{max\{f_{\phi'}(m') : m' =_{(x)} m\}, max\{f_{\phi''}(m') : m' =_{(x)} m\}\} = max\{max\{f_{\phi'}(m'), f_{\phi''}(m')\} : m' =_{(x)} m\} =$ (again by definition of validity of $\vee$) $max\{f_{\phi' \vee \phi''}(m') : m' =_{(x)} m\} = max\{f_\phi(m') : m' =_{(x)} m\} = f_{(\phi_{(x)})}(m)$.

(2) We have to show $f_{\phi_{(x)}} = f_{(\phi'_{(x)} \wedge \phi''_{(x)})}$. By definition of validity of $\wedge$, we have $f_{(\phi'_{(x)} \wedge \phi''_{(x)})}(m) = inf\{f_{\phi'_{(x)}}(m), f_{\phi''_{(x)}}(m)\}$. So $f_{(\phi'_{(x)} \wedge \phi''_{(x)})}(m) = inf\{max\{f_{\phi'}(m') : m' =_{(x)} m\}, max\{f_{\phi''}(m') : m' =_{(x)} m\}\} =$ (as ϕ'' does not contain x) $inf\{max\{f_{\phi'}(m') : m' =_{(x)} m\}, f_{\phi''}(m)\} = max\{inf\{f_{\phi'}(m'), f_{\phi''}(m)\} : m' =_{(x)} m\} =$ (again by definition of validity of $\wedge$, and by the fact that ϕ'' does not contain x) $max\{f_{\phi' \wedge \phi''}(m') : m' =_{(x)} m\} = max\{f_\phi(m') : m' =_{(x)} m\} = f_{(\phi\ )}(m)$.

□

Thus, we can calculate disjunctions separately, and also conjunctions, as long as the latter have no variables in common. In classical logic, we are finished, as we can break down conjunctions into parts which have no variables in common. The problem here are formulas of the type $a \to b$, as they may have variables in common with other conjuncts, and, as we will see in Fact 4.4.15 (page 162) (2) and (3), they cannot be eliminated.

Thus, we have to consider situations like $(a \to b) \wedge (b \to c)$, $a \wedge (a \to b)$, etc., where without loss of generality none is of the form $a \to a$, as this can be replaced by TRUE.

To do as many cases together as possible, it is useful to use Fact 4.4.5 (page 150) (9) and (11) backwards, to obtain general formulas. We then see that the cases to examine are of the form:

$\phi = ((b_1 \vee \ldots \vee b_n) \to a) \wedge (a \to (c_1 \wedge \ldots \wedge c_m)) \wedge \sigma a \wedge \tau a \wedge \rho a$, where none of the b_i or c_i are a, and where n, m may be 0, and σ, τ, ρ are absence ($\emptyset$, no a), a, $\neg a$, or $\neg\neg a$.

We have the following equalities (for $F = FALSE$, $T = TRUE$) :

$a \wedge \neg a = F$, $a \wedge \neg\neg a = a$, $\neg a \wedge \neg\neg a = F$, $a \wedge \neg a \wedge \neg\neg a = F$.

Thus, it suffices to consider σ as empty, a, $\neg a$, $\neg\neg a$, which leaves us with 4 cases. Moreover, we see that we always treat $b_1 \vee \ldots \vee b_n$ and $c_1 \wedge \ldots \wedge c_m$ as one block, so we can without loss of generality restrict the consideration to the 12 cases:

$\phi_{1,1} := (b \to a)$

$\phi_{1,2} := (b \to a) \wedge a$

$\phi_{1,3} := (b \to a) \wedge \neg a$

$\phi_{1,4} := (b \to a) \wedge \neg\neg a$

$\phi_{2,1} := (a \to c)$

Table 4.1: Table Neglecting a variable - Part 1

$\phi_{1,3}$		$b \to a$	$\neg a$	$(b \to a) \wedge \neg a$
	$a < b$	a		F
	$a = b$	T	$\neg b$	$\neg b$
	$a > b$	T	F	F
$\phi_{2,2}$		$a \to c$	a	$(a \to c) \wedge a$
	$a < c$	T	a	a
	$a = c$	T	c	c
	$a > c$	c	a	c
$\phi_{2,4}$		$a \to c$	$\neg\neg a$	$(a \to c) \wedge \neg\neg a$
	$a < c$	T	$\neg\neg a \leq \neg\neg c$	$\neg\neg a$
	$a = c$	T	$\neg\neg c$	$\neg\neg c$
	$a > c$	c	$\neg\neg a$ $(\geq \neg\neg c, c)$	c $(\leq \neg\neg c)$

$\phi_{2,2} := (a \to c) \wedge a$

$\phi_{2,3} := (a \to c) \wedge \neg a$

$\phi_{2,4} := (a \to c) \wedge \neg\neg a$

$\phi_{3,1} := (b \to a) \wedge (a \to c)$

$\phi_{3,2} := (b \to a) \wedge (a \to c) \wedge a$

$\phi_{3,3} := (b \to a) \wedge (a \to c) \wedge \neg a$

$\phi_{3,4} := (b \to a) \wedge (a \to c) \wedge \neg\neg a$

We consider now the maximum, when we let a float, i.e., consider all m' such that $m \restriction L - \{a\} = m' \restriction L - \{a\}$. Let $\phi'_{i,j}$ be this maximum. For $\phi_{1,1}$, $\phi_{1,2}$, $\phi_{1,4}$, $\phi'_{i,j} = T$ (take $a = T$).

For $\phi_{2,1}$, $\phi_{2,3}$, $\phi'_{i,j} = T$ (take $a = F$).

Next, we consider the remaining simple cases $\phi_{1,i}$ and $\phi_{2,i}$. We show $\phi'_{1,3} = \neg b$, $\phi'_{2,2} = c$, $\phi'_{2,4} = \neg\neg c$, see Table 4.4.2 (page 155). (We abbreviate e.g. $m(a) < m(b)$ by $a < b$.)

.

We show now $\phi'_{3,1} = b \to c$, $\phi'_{3,2} = c$, $\phi'_{3,3} = (b \to c) \wedge \neg b$, $\phi'_{3,4} = (b \to c) \wedge \neg\neg c$, see Table 4.4.2 (page 156).

Remark 4.4.9

We cannot improve the value of $\phi \to \psi$ by taking a detour $\phi \to \alpha_1 \to \ldots \to \alpha_n \to \psi$ because the destination determines the value: in any column of $\to$, there is only max and a constant value. And if we go further down than needed, we get only worse, going from right to left deteriorates the values in the lines. $\square$

We can achieve the same result by first closing under the following rules, and then erasing all formulas containing a :

Table 4.2: Table Neglecting a variable - Part 2

$\phi_{3,1}$		$b \to a$	$a \to c$		$(b \to a) \wedge (a \to c)$	$b \to c$		
	Case 1: $b \leq c$							
	1.1: $a < b$	a	T		a	T		
	1.2: $a = b$	T	T		T			
	1.3: $b \leq a \leq c$	T	T		T			
	1.4: $c < a$	T	c		c			
	Case 2: $c < b$							
	2.1: $a \leq c$	a	T		a $(\leq c)$	c		
	2.2: $c < a < b$	a	c		c			
	2.3: $b \leq a$	T	c		c			
$\phi_{3,2}$		$b \to a$	$a \to c$	a	$(b \to a) \wedge (a \to c) \wedge a$	c		
	Case 1: $b \leq c$							
	1.1: $a < b$	a	T		a $(< c)$			
	1.2: $a = b$	T	T		a $(\leq c)$			
	1.3: $b \leq a \leq c$	T	T		a $(\leq c)$			
	1.4: $c < a$	T	c		c			
	Case 2: $c < b$							
	2.1: $a \leq c$	a	T		a $(\leq c)$			
	2.2: $c < a < b$	a	c		c			
	2.3: $b \leq a$	T	c		c			
$\phi_{3,3}$		$b \to a$	$a \to c$	$\neg a$	$(b \to a) \wedge (a \to c) \wedge \neg a$	$b \to c$	$\neg b$	$(b \to c) \wedge \neg b$
	Case 1: $b \leq c$							
	1.1: $a < b$	a	T	$\neg a$	F	T	$\neg b$	$\neg b$
	1.2: $a = b$	T	T	$\neg a$	$\neg a = \neg b$			
	1.3: $b \leq a \leq c$	T	T	$\neg a$ $(\leq \neg b)$	$\neg a$			
	1.4: $c < a$	T	c	$\neg a$ $(\leq \neg b, \neg c)$	$\neg a \wedge c = F$			
	Case 2: $c < b$							
	2.1: $a \leq c$	a	T	$\neg a$	F	c	$\neg b = F$	F
	2.2: $c < a < b$	a	c	$\neg a$	F			
	2.3: $b \leq a$	T	c	$\neg a$ $(\leq \neg c)$	F			
$\phi_{3,4}$		$b \to a$	$a \to c$	$\neg\neg a$	$(b \to a) \wedge (a \to c) \wedge \neg\neg a$	$b \to c$	$\neg\neg c$	$(b \to c) \wedge \neg\neg c$
	Case 1: $b \leq c$							
	1.1: $a < b$	a	T	$\neg\neg a$	$a \leq \neg\neg c$	T	$\neg\neg c$	$\neg\neg c$
	1.2: $a = b$	T	T	$\neg\neg a$	$\neg\neg a \leq \neg\neg c$			
	1.3: $b \leq a \leq c$	T	T	$\neg\neg a$	$\neg\neg a \leq \neg\neg c$ $= \neg\neg c$ if $a = c$			
	1.4: $c < a$	T	c	T	$c \leq \neg\neg c$			
	Case 2: $c < b$							
	2.1: $a \leq c$	a	T	$\neg\neg a$	$a \leq c$	c	$\neg\neg c$	c
	2.2: $c < a < b$	a	c	T	c			
	2.3: $b \leq a$	T	c	T	c			

(1) $\rightarrow$ under transitivity, i.e.

$((b_1 \vee \ldots \vee b_n) \rightarrow a) \wedge (a \rightarrow (c_1 \wedge \ldots \wedge c_m)) \Rightarrow ((b_1 \vee \ldots \vee b_n) \rightarrow (c_1 \wedge \ldots \wedge c_m))$

(2) $\sigma' a$ and $\rightarrow$ as follows:

$((b_1 \vee \ldots \vee b_n) \rightarrow a) \wedge (a \rightarrow (c_1 \wedge \ldots \wedge c_m)),\ a \Rightarrow ((b_1 \vee \ldots \vee b_n) \rightarrow (c_1 \wedge \ldots \wedge c_m)) \wedge c_1 \wedge \ldots \wedge c_m$

$((b_1 \vee \ldots \vee b_n) \rightarrow a) \wedge (a \rightarrow (c_1 \wedge \ldots \wedge c_m)),\ \neg\neg a \Rightarrow ((b_1 \vee \ldots \vee b_n) \rightarrow (c_1 \wedge \ldots \wedge c_m)) \wedge \neg\neg c_1 \wedge \ldots \wedge \neg\neg c_m$

$((b_1 \vee \ldots \vee b_n) \rightarrow a) \wedge (a \rightarrow (c_1 \wedge \ldots \wedge c_m)),\ \neg a \Rightarrow ((b_1 \vee \ldots \vee b_n) \rightarrow (c_1 \wedge \ldots \wedge c_m)) \wedge \neg b_1 \wedge \ldots \wedge \neg b_n$

In summary: the semantical interpolant constructed in Section 2.2.2.3 (page 51) is definable, if the assumption holds, so the HT logic (see Section 4.4.4 (page 162)) has also syntactic interpolation. This result is well-known, but we need the techniques for the next section.

In some cases, introducing new constants analogous to TRUE, FALSE - in the cited case e.g. ONE, TWO when truth starts at world one or two - might help, but we did not investigate this question. This question is also examined in [ABM03].

4.4.3.2 An important example for non-existence of interpolation

We turn now to an important example. It shows that the logic with 3 worlds, and thus 4 truth values, has no interpolation. But first, we show as much as possible for the general case (arbitrarily many truth values).

Example 4.4.1

Let

$\alpha(p, q, r) := \Big(p \rightarrow (((q \rightarrow r) \rightarrow q) \rightarrow q)\Big) \rightarrow p,$

$\beta(p, s) := ((s \rightarrow p) \rightarrow s) \rightarrow s$

We will show that $\alpha(p, q, r) \rightarrow \beta(p, s)$ holds in the case of 3 worlds, but that there is no syntactic interpolant (which could use only p).

Introducing a new operator Jp meaning "from next moment onwards p holds and if now is the last moment then p holds now" gives enough definability to have also syntactic interpolation for α and β above. This will be shown in Section 4.4.5 (page 163). First, we give some general results for above example.

Fact 4.4.10

Let T, truth, be the maximal truth value.

(1) $\phi := ((\ldots(((a \rightarrow b) \rightarrow b) \rightarrow b)\ldots) \rightarrow b)$ has the following truth value $v(\phi)$ in a model m :

- (1.1) if the number n of b on the right of the first $\rightarrow$ is odd:
  if $v(a) \leq v(b)$, then $v(\phi) = T$, otherwise $v(\phi) = v(b)$,
- (1.2) if the number m of b on the right of the first $\rightarrow$ is even:
  if $v(a) \leq v(b)$, then $v(\phi) = v(b)$, otherwise $v(\phi) = T$.

(2) $\phi := ((\ldots(((a \rightarrow b) \rightarrow a) \rightarrow a)\ldots) \rightarrow a)$ has the following truth value $v(\phi)$ in a model m :

(2.1) if the number n of a on the right of the first $\rightarrow$ is odd:
if $v(b) < v(a)$, then $v(\phi) = T$, otherwise $v(\phi) = v(a)$,

(2.2) if the number m of a on the right of the first $\rightarrow$ is even:
if $v(b) < v(a)$, then $v(\phi) = v(a)$, otherwise $v(\phi) = T$.

Proof

(1)

We proceed by induction.

(1.1) For $n = 1$, it is the definition of $\rightarrow$.

(1.2) Case $n = 2$: If $v(a) \leq v(b)$, then $v(a \rightarrow b) = T$, so $v((a \rightarrow b) \rightarrow b) = v(b)$. If $v(a) > v(b)$, then $v(a \rightarrow b) = v(b)$, so $v((a \rightarrow b) \rightarrow b) = T$.

The general induction works as for the step from $n = 1$ to $n = 2$.

(2)

$n = 1$:

$\phi = (a \rightarrow b) \rightarrow a$. If $v(b) < v(a)$, then $v(a \rightarrow b) = v(b)$, so $v(\phi) = T$. If $v(b) \geq v(a)$, then $v(a \rightarrow b) = T$, so $\phi(\phi) = v(a)$.

$n \rightarrow n + 1$:

$\phi = \psi \rightarrow a$. If $v(b) < v(a)$, then, if n is odd, $v(\psi) = T$, so $v(\phi) = v(a)$, if n is even, $v(\psi) = v(a)$, so $v(\phi) = T$. If $v(b) \geq v(a)$, then, if n is odd, $v(\psi) = v(a)$, so $v(\phi) = T$, if n is even, $v(\psi) = T$, so $v(\phi) = v(a)$.

□

Corollary 4.4.11

Let T be the maximal truth value TRUE.

Consider again the formulas of Example 4.4.1 (page 157), $\alpha(p, q, r) := \Big(p \rightarrow (((q \rightarrow r) \rightarrow q) \rightarrow q)\Big) \rightarrow p$, $\beta(p, s) := ((s \rightarrow p) \rightarrow s) \rightarrow s$.

We use Fact 4.4.10 (page 157) , the numbers refer to this fact.

Let $f := f_\beta$. Let $f'(m) := min\{f(m') : m \upharpoonright p = m' \upharpoonright p\}$. Fix m. By (2.2), if $m(p) = T$, then $f'(m) = T$, if $m(p) < T$, then $f'(m) = m(p) + 1$.

Let $g := f_\alpha$. Let $g'(m) := max\{g(m') : m \upharpoonright p = m' \upharpoonright p\}$. Fix m. α is of the form $(p \rightarrow \phi) \rightarrow p$, so by (2.1), if $m'(\phi) < m'(p)$, then $m'(\alpha) = T$, if $m'(\phi) \geq m'(p)$, then $m'(\alpha) = m'(p)$. By (2.2), we have: if $m'(r) < m'(q)$, then $m'(\phi) = m'(q)$, if $m'(r) \geq m'(q)$, then $m'(\phi) = T$. Note that $m'(\phi) > 0$.

Table 4.4.3.2 (page 159) shows that for $T = 3$ $\alpha \vdash \beta$, for $T = 4$ $\alpha \not\vdash \beta$.

Thus, an interpolant h must have $h(0) = 0$ or 1, $h(1) = 1$ or 2, $h(2) = h(3) = 3$ in the case $T = 3$. This is impossible by Fact 4.4.5 (page 150).

□

Table 4.3: Table $\alpha \vdash \beta$

T=3			T=4		
$m(p)$	$f'(m)$	$g'(m)$	$m(p)$	$f'(m)$	$g'(m)$
0	1	0	0	1	0
1	2	1	1	2	1
2	3	3	2	3	4
3	3	3	3	4	4
			4	4	4

.

4.4.3.3 The additional operators J and A

The following was checked with a small computer program:

(1) A alone will generate 12 semantically different formulas with 1 variable, but it does not suffice to obtain interpolation.

(2) J alone will generate 8 semantically different formulas with 1 variable, and it will solve the interpolation problem for $\alpha(p,q,r)$ and $\beta(p,s)$ of Example 4.4.1 (page 157)

(3) A and J will generate 48 semantically different formulas with 1 variable.

4.4.3.4 The additional operator F

For $n+1$ truth values, let for $k < n$

$\phi'_k := \neg(F^k(a) \rightarrow F^{k+1}(a))$.

$\phi_k := a \wedge \phi'_k$.

(For $k = n$, we take $\neg a$.)

Then

$$f_{\phi_k}(m) := \begin{cases} n-k & iff \quad m = n-k \\ FALSE & \quad otherwise \end{cases}$$

Applying F again, we can increase the value from $n-k$ up to TRUE.

We give the table of $\phi'_k(\alpha)$ for 6 truth values.

α	0	1	2	3	4	5
$\phi'_5 = \neg\alpha$	5	0	0	0	0	0
$\phi'_4 = \neg(F^4(\alpha) \rightarrow F^5(\alpha))$	0	5	0	0	0	0
$\phi'_3 = \neg(F^3(\alpha) \rightarrow F^4(\alpha))$	0	0	5	0	0	0
$\phi'_2 = \neg(F^2(\alpha) \rightarrow F^3(\alpha))$	0	0	0	5	0	0
$\phi'_1 = \neg(F(\alpha) \rightarrow F^2(\alpha))$	0	0	0	0	5	0
$\phi'_0 = \neg(\alpha \rightarrow F(\alpha))$	0	0	0	0	0	5

This allows definition by cases. Suppose we want α to have the same result as ψ if ψ has value p, and the same result as ψ' otherwise, more precisely:

$$F_\alpha(m) := \begin{cases} F_\psi(m) & iff \quad F_\psi(m) = p \\ F_{\psi'}(m) & otherwise \end{cases}$$

then we define:

$\alpha := (\phi'_{n-p}(\psi) \wedge \psi) \vee ((\neg\phi'_{n-p}(\psi)) \wedge \psi')$.

As ϕ_k contains $\rightarrow$, but only 1 variable, there is no problem for projections here.

Note the following important fact:

More generally, we can construct in the same way new functions from old ones by cases, like:

$$F_\psi(m) := \begin{cases} F_\sigma(m) & iff \quad F_\alpha(m) = s \\ F_\tau(m) & iff \quad F_\alpha(m) = t \\ F_\rho(m) & otherwise \end{cases}$$

But we can *not* attribute arbitrary values as in

if condition1 holds, then x1, if condition2 holds, then x2, etc.

This is also reflected by the fact that by the above, for 4 truth values, we can (using $\vee$) obtain $\{0,3\} \times \{0,1,2,3\} \times \{0,2,3\} \times \{0,3\} = 2*4*3*2 = 48$ semantically different formulas with 1 variable, and no more (checked with a computer program). Therefore, F is weaker than Z, which can generate arbitrary functions.

Thus, we can also define $\sigma \rightarrow \tau$ by cases, in a uniform way for all n. Consider, e.g., the case $v(\phi) = 3$, $v(\psi) = 2$. $v(\phi \rightarrow \psi)$ should be equal to $v(\psi)$, we take: $\phi_{n-3}(\phi) \wedge \phi_{n-2}(\psi) \wedge \psi$.

We conclude by the following

Fact 4.4.12

F and $J + A$ are interdefinable:

$Aa \leftrightarrow \neg(a \rightarrow Fa)$,

$Ja \leftrightarrow Fa \vee Aa$,

$Fa \leftrightarrow Ja \wedge \neg Aa.$

□

4.4.3.5 The additional operator Z

We turn to the operator Z.

Definition 4.4.4

We introduce the following, derived, auxiliary, operators:

$S_i(\phi) := n$ iff $v(\phi) = i$, and 0 otherwise, for $i = 0, \ldots, n$

$K_i(\phi) := i$ for any ϕ, for $i = 0, \ldots, n$

Example 4.4.2

We give here the example for $n = 5$.

	a	0	1	2	3	4	5
Z		1	2	3	4	5	0
	a	0	1	2	3	4	5
K_0		0	0	0	0	0	0
K_1		1	1	1	1	1	1
K_2		2	2	2	2	2	2
K_3		3	3	3	3	3	3
K_4		4	4	4	4	4	4
K_5		5	5	5	5	5	5
	a	0	1	2	3	4	5
S_0		5	0	0	0	0	0
S_1		0	5	0	0	0	0
S_2		0	0	5	0	0	0
S_3		0	0	0	5	0	0
S_4		0	0	0	0	5	0
S_5		0	0	0	0	0	5

Fact 4.4.13

(1) We can define $S_i(\phi)$ and $K_i(\phi)$ for $0 \leq i \leq n$ from $\neg, \wedge, \vee, Z$.

(2) We can define any $m-$ary truth function from $\neg, \wedge, \vee, Z$.

Proof

(1)

$S_i(\phi) = \neg Z^{n-i}(\phi)$, $K_i(\phi) = Z^{i+1}(\neg(\phi \wedge \neg\phi))$

(2)

Suppose $\langle i_1, \ldots, i_m \rangle$ should have value i, $\langle i_1, \ldots, i_m \rangle \mapsto i$, we can express this by

$S_{i_1}(x_1) \wedge, \ldots, \wedge S_{i_m}(x_m) \wedge K_i$.

We then take the disjunction of all such expressions:

$\bigvee \{S_{i_1}(x_1) \wedge, \ldots, \wedge S_{i_m}(x_m) \wedge K_i : \langle i_1, \ldots, i_m \rangle \mapsto i\}$

Corollary 4.4.14

Any model function is definable from Z, so any semantical interpolant is also a syntactic one. $\Box$

4.4.4 The three valued intuitionistic logic Here/There HT

We now give a short introduction to the well-known 3-valued intuitionistic logic HT (Here/There), with some results also for similar logics with more than 3 values. Many of these properties were found and checked with a small computer program. In particular, we show the existence of a normal form, similar to classical propositional logic, but $\rightarrow$ cannot be eliminated. Consequently, we cannot always separate propositional variables easily.

Our main result here (which is probably well known, we claim no priority) is that "forgetting" a variable preserves definability in the following sense: Let, e.g., $\phi = a \wedge b$, and $M(\phi)$ be the set of models where ϕ has maximal truth value (2 here), then there is ϕ' such that the set of models where ϕ' has value 2 is the set of all models which agree with a model of ϕ on, e.g., b. We "forget" about a. Our ϕ' is here, of course, b. Here, the problem is trivial, it is a bit less so when $\rightarrow$ is involved, as we cannot always separate the two parts. For example, the result of "forgetting" about a in the formula $a \wedge (a \rightarrow b)$ is b, in the formula $a \rightarrow b$ it is TRUE. Thus, forgetting about a variable preserves definability, and the abovementioned semantical interpolation property carries over to the syntactic side, similarly to the result on classical logic, see Section 4.3.3.3 (page 139) .

Fact 4.4.15

These results were checked with a small computer program:

(1) With 2 variables a, b are definable, using the operators $\neg$, $\rightarrow$, $\wedge$, $\vee$, 174 semantically different formulas. $\vee$ is not needed, i.e. with or without $\vee$ we have the same set of definable formulas. We have, e.g., $a \vee b \leftrightarrow \Big(\big(b \rightarrow (\neg\neg a \rightarrow a)\big) \rightarrow \big((\neg a \rightarrow b) \wedge (\neg\neg a \rightarrow a)\big)\Big)$.

(2) With the operators $\neg$, $\wedge$, $\vee$ only 120 semantically different formulas are definable. Thus, $\rightarrow$ cannot be expressed by the other operators.

Fact 4.4.16

$((a \rightarrow b) \rightarrow c) \leftrightarrow ((\neg a \rightarrow c) \wedge (b \rightarrow c) \wedge (a \vee \neg b \vee c))$

holds in the 3 valued case.

(Thanks to D.Pearce for telling us.)

This Fact is well known, we have verified it by computer, but not by hand.

Corollary 4.4.17

The 3 valued case has interpolation.

Proof

By Fact 4.4.16 (page 162), we can flatten nested $\rightarrow$'s, so we have projection. $\square$

4.4.5 Finite Goedel logics with 4 truth values

Interpolation fails for:

$(d \rightarrow (((a \rightarrow b) \rightarrow a) \rightarrow a) \rightarrow d \vdash (((c \rightarrow d) \rightarrow c) \rightarrow c.$

Consider the table in Corollary 4.4.11 (page 158), and the comment about possible interpolants after the table. By the proof of Fact 4.4.5 (page 150) , (0), we see that above formulas have no interpolant. But it is trivial to see that Jp will be an interpolant, see Definition 4.4.3 (page 149).

Note that the implication is not true for more than 4 truth values, as we saw in Corollary 4.4.11 (page 158), so this example will not be a counterexample to interpolation any more.

We have checked with a computer program, but not by hand:

Introducing a new constant, 1, which has always truth value 1, and is thus simpler than above operator J, gives exactly 2 different interpolants for the formulas of Example 4.4.1 (page 157): $(p \rightarrow 1) \rightarrow p$, and $(p \rightarrow 1) \rightarrow 1$. Introducing an additional constant 2 will give still other interpolants. But if one is permitted in classical logic to use the constants TRUE and FALSE for interpolation, why not 1 and 2 here?

Chapter 5

Laws about size and interpolation in non-monotonic logics

5.1 Introduction

5.1.1 Various concepts of size and non-monotonic logics

A natural interpretation of the non-monotonic rule $\phi \mid\sim \psi$ is that the set of exceptional cases, i.e., those where ϕ holds, but not ψ, is a small subset of all the cases where ϕ holds, and the complement, i.e., the set of cases where ϕ and ψ hold, is a big subset of all $\phi-$cases.

This interpretation gives an abstract semantics to non-monotonic logic, in the sense that definitions and rules are translated to rules about model sets, without any structural justification of those rules, as they are given, e.g., by preferential structures, which provide structural semantics. Yet, they are extremely useful, as they allow us to concentrate on the essentials, forgetting about syntactical reformulations of semantically equivalent formulas, the laws derived from the standard proof theoretical rules incite to generalizations and modifications, and reveal deep connections but also differences. One of those insights is the connection between laws about size and (semantical) interpolation for non-monotonic logics.

To put this abstract view a little more into perspective, we present three alternative systems, also working with abstract size as a semantics for non-monotonic logics. (They were already mentioned in Section 1.5.4 (page 24) .)

- the system of one of the authors for a first order setting, published in [Sch90] and elaborated in [Sch95-1],
- the system of S.Ben-David and R.Ben-Eliyahu, published in [BB94],
- the system of N.Friedman and J.Halpern, published in [FH96].

(1) Defaults as generalized quantifiers:

We first recall the definition of a "weak filter", made official in Definition 2.2.3 (page 42) :

Fix a base set X. A weak filter on or over X is a set $\mathcal{F} \subseteq \mathcal{P}(X)$, s.t. the following conditions hold:

(F1) $X \in \mathcal{F}$

(F2) $A \subseteq B \subseteq X$, $A \in \mathcal{F}$ imply $B \in \mathcal{F}$

$(F3')$ $A, B \in \mathcal{F}$ imply $A \cap B \neq \emptyset$.

We use weak filters on the semantical side, and add the following axioms on the syntactical side to a FOL axiomatisation:

1. $\nabla x \phi(x) \wedge \forall x(\phi(x) \rightarrow \psi(x)) \rightarrow \nabla x \psi(x)$,
2. $\nabla x \phi(x) \rightarrow \neg \nabla x \neg \phi(x)$,
3. $\forall x \phi(x) \rightarrow \nabla x \phi(x)$ and $\nabla x \phi(x) \rightarrow \exists x \phi(x)$.

A model is now a pair, consisting of a classical FOL model M, and a weak filter over its universe. Both sides are connected by the following definition, where $\mathcal{N}(M)$ is the weak filter on the universe of the classical model M :

$\langle M, \mathcal{N}(M)\rangle \models \nabla x \phi(x)$ iff there is $A \in \mathcal{N}(M)$ s.t. $\forall a \in A$ $(\langle M, \mathcal{N}(M)\rangle \models \phi[a])$.

Soundness and completeness is shown in [Sch95-1], see also [Sch04].

The extension to defaults with prerequisites by restricted quantifiers is straightforward.

(2) The system of $S.$ Ben-David and $R.$ Ben-Eliyahu:

Let $\mathcal{N}' := \{\mathcal{N}'(A) : A \subseteq U\}$ be a system of filters for $\mathcal{P}(U)$, i.e. each $\mathcal{N}'(A)$ is a filter over A. The conditions are (in slight modification):

UC': $B \in \mathcal{N}'(A) \rightarrow \mathcal{N}'(B) \subseteq \mathcal{N}'(A)$,

DC': $B \in \mathcal{N}'(A) \rightarrow \mathcal{N}'(A) \cap \mathcal{P}(B) \subseteq \mathcal{N}'(B)$,

RBC': $X \in \mathcal{N}'(A)$, $Y \in \mathcal{N}'(B) \rightarrow X \cup Y \in \mathcal{N}'(A \cup B)$,

SRM': $X \in \mathcal{N}'(A)$, $Y \subseteq A \rightarrow A - Y \in \mathcal{N}'(A) \vee X \cap Y \in \mathcal{N}'(Y)$,

GTS': $C \in \mathcal{N}'(A)$, $B \subseteq A \rightarrow C \cap B \in \mathcal{N}'(B)$.

(3) The system of $N.$ Friedman and $J.$ Halpern:

Let U be a set, $<$ a strict partial order on $\mathcal{P}(U)$, (i.e. $<$ is transitive, and contains no cycles). Consider the following conditions for $<$:

(B1) $A' \subseteq A < B \subseteq B' \rightarrow A' < B'$,

(B2) if A, B, C are pairwise disjoint, then $C < A \cup B$, $B < A \cup C \rightarrow B \cup C < A$,

(B3) $\emptyset < X$ for all $X \neq \emptyset$,

(B4) $A < B \rightarrow A < B-$A,

(B5) Let $X, Y \subseteq A$. If $A - X < X$, then $Y < A - Y$ or $Y - X < X \cap Y$.

The equivalence of the systems of [BB94] and [FH96] was shown in [Sch97-4], see also [Sch04].

Historical remarks: Our own view as abstract size was inspired by the classical filter approach, as used e.g. in mathematical measure theory. The first time that abstract size was related to nonmonotonic logics was, to our knowledge, in the second author's [Sch90] and [Sch95-1], and, independently, in [BB94]. The approach to size by partial orders is first discussed - to our knowledge

- by N.Friedman and J.Halpern, see [FH96]. More detailed remarks can also be found in [GS08c], [GS09a], [GS08f]. A somewhat different approach is taken in [HM07].

Before we introduce the connection between interpolation and multiplicative laws about size, we give now some comments on the laws about size themselves.

5.1.2 Additive and multiplicative laws about size

We give here a short introduction to and some examples for additive and multiplicative laws about size. A detailed overview is presented in Table 5.1 (page 189), Table 5.2 (page 190), and Table 5.3 (page 191). (The first two tables have to be read together, they are too big to fit on one page.)

They show connections and how to develop a multitude of logical rules known from nonmonotonic logics by combining a small number of principles about size. We can use them as building blocks to construct the rules from. More precisely, "size" is to be read as "relative size", since it is essential to change the base sets.

In the first two tables, these principles are some basic and very natural postulates, (Opt), (iM), $(eM\mathcal{I})$, $(eM\mathcal{F})$, and a continuum of power of the notion of "small", or, dually, "big", from $(1 * s)$ to $(< \omega * s)$. From these, we can develop the rest except, essentially, Rational Monotony, and thus an infinity of different rules.

The probably easiest way to see a connection between non-monotonic logics and abstract size is by considering preferential structures. Preferential structures define principal filters, generated by the set of minimal elements, as follows: if $\phi \mid\sim \psi$ holds in such a structure, then $\mu(\phi) \subseteq M(\psi)$, where $\mu(\phi)$ is the set of minimal elements of $M(\phi)$. According to our ideas, we define a principal filter $\mathcal{F}$ over $M(\phi)$ by $X \in \mathcal{F}$ iff $\mu(\phi) \subseteq X \subseteq M(\phi)$. Thus, $M(\phi) \cap M(\neg\psi)$ will be a "small" subset of $M(\phi)$. (Recall that filters contain the "big" sets, and ideals the "small" sets.)

We can now go back and forth between rules on size and logical rules, e.g.:

(For details, see Table 5.1 (page 189), Table 5.2 (page 190), and Table 5.3 (page 191).)

(1) The "AND" rule corresponds to the filter property (finite intersections of big subsets are still big).

(2) "Right weakening" corresponds to the rule that supersets of big sets are still big.

(3) It is natural, but beyond filter properties themselves, to postulate that, if X is a small subset of Y, and $Y \subseteq Y'$, then X is also a small subset of Y'. We call such properties "coherence properties" between filters. This property corresponds to the logical rule (wOR).

(4) In the rule (CM_ω), usually called Cautious Monotony, we change the base set a little when going from $M(\alpha)$ to $M(\alpha \wedge \beta)$ (the change is small by the prerequisite $\alpha \mid\sim \beta$), and still have $\alpha \wedge \beta \mid\sim \beta'$, if we had $\alpha \mid\sim \beta'$. We see here a conceptually very different use of "small", as we now change the base set, over which the filter is defined, by a small amount.

(5) The rule of Rational Monotony is the last one in the first table, and somewhat isolated there. It is better to be seen as a multiplicative law, as described in the third table. It corresponds to the rule that the product of medium (i.e, neither big nor small) sets, has still medium size.

5.1.3 Interpolation and size

The connection between non-monotonic logic and the abstract concept of size was investigated in [GS09a], see also [GS08f]. There, we looked among other things at abstract addition of size. Here, we will show a connection to abstract multiplication of size. Our semantic approach used decomposition of set theoretical products. An important step was to write a set of models Σ as a product of some set Σ' (which was a restriction of Σ), and some full Cartesian product. So, when we speak about size, we will have (slightly simplified) some big subset Σ_1 of one product Π_1, and some big subset Σ_2 of another product Π_2, and will now check whether $\Sigma_1 \times \Sigma_2$ is a big subset of $\Pi_1 \times \Pi_2$. In shorthand, whether "$big * big = big$". (See Definition 5.2.1 (page 178) for precise definitions.) Such conditions are called coherence conditions, as they do not concern the notion of size itself, but the way the sizes defined for different base sets are connected. Our main results here are Proposition 5.3.3 (page 197) and Proposition 5.3.5 (page 198). They say that if the logic under investigation is defined from a notion of size which satisfies sufficiently many conditions, then this logic will have interpolation of type one or even two.

Consider now some set product $X \times X'$. (Intuitively, X and X' are model sets on sublanguages J and J' of the whole language L.) When we have now a rule like: If Y is a big subset of X, and Y' a big subset of X', then $Y \times Y'$ is a big subset of $X \times X'$, and conversely, we can calculate consequences separately in the sublanguages, and put them together to have the overall consequences. But this is the principle behind interpolation: we can work with independent parts.

This is made precise in Definition 5.2.1 (page 178), in particular by the rule

$(\mu * 1) : \mu(X \times X') = \mu(X) \times \mu(X')$.

(Note that the conditions $(\mu * i)$ and $(\Sigma * i)$ are equivalent, as shown in Proposition 5.2.1 (page 179) (for principal filters).)

The main result is that the multiplicative size rule $(\mu * 1)$ entails non-monotonic interpolation of the form $\phi \mid\sim \alpha \mid\sim \psi$, see Proposition 5.3.5 (page 198).

We take now a closer look at interpolation for non-monotonic logic.

The three variants of interpolation

Consider preferential logic, a rule like $\phi \mid\sim \psi$. This means that $\mu(\phi) \subseteq M(\psi)$. So we go from $M(\phi)$ to $\mu(\phi)$, the minimal models of ϕ, and then to $M(\psi)$, and, abstractly, we have $M(\phi) \supseteq \mu(\phi) \subseteq M(\psi)$, so we have neither necessarily $M(\phi) \subseteq M(\psi)$, nor $M(\phi) \supseteq M(\psi)$, the relation between $M(\phi)$ and $M(\psi)$ may be more complicated. Thus, we have neither the monotone, nor the antitone case. For this reason, our general results for monotone or antitone logics do not hold any more.

But we also see here that classical logic is used, too. Suppose that there is ϕ' which describes exactly $\mu(\phi)$, then we can write $\phi \mid\sim \phi' \vdash \psi$.

So we can split preferential logic into a core part - going from ϕ to its minimal models - and a second part, which is just classical logic. (Similar decompositions are also natural for other non-monotonic logics.) Thus, preferential logic can be seen as a combination of two logics, the non-monotonic core, and classical logic. It is thus natural to consider variants of the interpolation problem, where $\mid\sim$ denotes again preferential logic, and $\vdash$ as usual classical logic:

Given $\phi \mid\sim \psi$, is there "simple" α such that

(1) $\phi \mid\!\sim \alpha \vdash \psi$, or

(2) $\phi \vdash \alpha \mid\!\sim \psi$, or

(3) $\phi \mid\!\sim \alpha \mid\!\sim \psi$?

In most cases, we will only consider the semantical version, as the problems of the syntactical version are very similar to those for monotonic logics. We turn to the variants.

(1) The first variant, $\phi \mid\!\sim \alpha \vdash \psi$, has a complete characterization in Proposition 5.3.2 (page 195), provided we have a suitable normal form (conjunctions of disjunctions). The condition says that the relevant variables of $\mu(\phi)$ have to be relevant for $M(\phi)$.

(2) The second variant, $\phi \vdash \alpha \mid\!\sim \psi$, is related to very (and in many cases, too) strong conditions about size. We do not have a complete characterization, only sufficient conditions about size. The size conditions we need are (see Definition 5.2.1 (page 178)):

the abovementioned $(\mu * 1)$, and,

$(\mu * 2) : \mu(X) \subseteq Y \Rightarrow \mu(X \upharpoonright A) \subseteq Y \upharpoonright A$

where X need not be a product any more.

The result is given in Proposition 5.3.3 (page 197).

Example 5.2.1 (page 181) shows that $(\mu * 2)$ seems too strong when compared to probability defined size.

We should, however, note that sufficiently modular preferential relations guarantee these very strong properties of the big sets, see Section 5.2.3 (page 181).

(3) We turn to the third variant, $\phi \mid\!\sim \alpha \mid\!\sim \psi$. This is probably the most interesting one, as it is more general, loosens the connection with classical logic, seems more natural as a rule, and is also connected to more natural laws about size. Again, we do not have a complete characterization, only sufficient conditions about size. Here, $(\mu * 1)$ suffices, and we have our main result about non-monotonic semanti interpolation, Proposition 5.3.5 (page 198), that $(\mu * 1)$ entails interpolation of the type $\phi \mid\!\sim \alpha \mid\!\sim \psi$.

Proposition 5.2.4 (page 182) shows that $(\mu * 1)$ is (roughly) equivalent to the properties

$(GH1)$ $\sigma \preceq \tau \wedge \sigma' \preceq \tau' \wedge (\sigma \prec \tau \vee \sigma' \prec \tau') \Rightarrow \sigma\sigma' \prec \tau\tau'$

(where $\sigma \preceq \tau$ iff $\sigma \prec \tau$ or $\sigma = \tau$)

$(GH2)$ $\sigma\sigma' \prec \tau\tau' \Rightarrow \sigma \prec \tau \vee \sigma' \prec \tau'$

of a preferential relation.

($(GH2)$ means that some compensation is possible, e.g., $\tau \prec \sigma$ might be the case, but $\sigma' \prec \tau'$ wins in the end, so $\sigma\sigma' \prec \tau\tau'$.)

There need not always be a semantical interpolation for the third variant, this is shown in Example 5.3.1 (page 193).

So we see that, roughly, semantic interpolation for nonmonotonic logics works when abstract size is defined in a modular way - and we see independence again. In a way, this is not surprising, as we use independent definition of validity for interpolation in classical logic, and we use independent definition of additional structure (relations or size) for interpolation in non-monotonic logic.

5.1.4 Hamming relations and size

As preferential relations are determined by a relation, and give rise to abstract notions of size and their manipulation, it is natural to take a close look at the corresponding properties of the relation. We already gave a few examples in the preceding sections, so we can be concise here. Our main definitions and results on this subject are to be found in Section 5.2.3 (page 181), where we also discuss distances with similar properties.

It is not surprising that we find various types of Hamming relations and distances in this context, as they are, by definition, modular. Neither is it surprising that we see them again in Chapter 6 (page 213), as we are interested there in independent ways to define neighbourhoods.

Basically, these relations and distances come in two flavours, the set and the counting variant. This is perhaps best illustrated by the Hamming distance of two sequence of finite, equal length. We can define the distance by the *set* of arguments where they differ, or by the *cardinality* of this set. The first results in possibly incomparable distances, the second allows "compensation", difference in one argument can be compensated by equality in another argument.

For definitions and results, also those connecting them to notions of size, see Section 5.2.3 (page 181) in particular Definition 5.2.2 (page 181). We then show in Proposition 5.2.4 (page 182) that (smooth) Hamming relations generate our size conditions when size is defined as above from a relation (the set of preferred elements generates the principal filter). Thus, Hamming relations determine logics which have interpolation, see Corollary 5.3.4 (page 198).

5.1.5 Equilibrium logic

Equilibrium logic, due to D.Pearce, A.Valverde, see [PV09] for motivation and further discussion, is based on the 3-valued finite Goedel logic, also called HT logic, HT for "here and there". Our results are presented in Section 5.3.6 (page 203).

Equilibrium logic (EQ) is defined by a choice function on the model set. First models have to be "total", no variable of the language may have 1 as value. Second, if $m \prec m'$, then m is considered better, and m' discarded, where $m \prec m'$ iff m and m' give value 0 to the same variables, and m gives value 2 to strictly less (as subset) variables than m' does.

We can define equilibrium logic by a preferential relation (taking care also of the first condition), but it is not smooth. Thus, our general results from the beginning of this section will not hold, and we have to work with "hand knitted" solutions. We first show that equilibrium logic has no interpolation of the form $\phi \vdash \alpha \mid\sim \psi$ or $\phi \mid\sim \alpha \vdash \psi$, then that is has interpolation of the form $\phi \mid\sim \alpha \mid\sim \psi$, and that the interpolant is also definable, i.e., equilibrium logic has semantic and syntactic interpolation of this form. Essentially, semantic interpolation is due to the fact that the preference relation is defined in a modular way, using individual variables - as always, when we have interpolation.

5.1.6 Interpolation for revision and argumentation

We have a short and simple result (Lemma 5.3.6 (page 203)) for interpolation in AGM revision. Unfortunately, we need the variables from both sides of the revision operator as can easily be seen by revising with TRUE. The reader is referred to Section 5.3.5 (page 203) for details.

Somewhat surprisingly, we also have an interpolation result for one form of argumentation, where we consider the set of arguments for a statement as the truth value of that statement. As we have maximum (set union), we have the lower bound used in Proposition 4.2.3 (page 134) for the monotonic case, and can show Fact 5.5.3 (page 211). See Section 5.5 (page 209) for details.

5.1.7 Language change to obtain products

To achieve interpolation and other results of independence, we often need to write a set of models as a non-trivial product. Sometimes, this is impossible, but an equivalent reformulation of the language can solve the problem, see Example 5.2.5 (page 192).

Crucial there is that $6 = 3 * 2$, so we can just re-arrange the 6 models in a different way, see Fact 5.2.9 (page 192).

A similar result holds for the non-monotonic case, where the structure must be possible, we can then redefine the language.

All details are to be found in Section 5.2.5 (page 192).

5.2 Laws about size

5.2.1 Additive laws about size

We now give the main additive rules for manipulation of abstract size from [GS09a], see Table 5.1 (page 189) and Table 5.2 (page 190), "Rules on size".

The notation is explained with some redundancy, so the reader will not have to leaf back and forth to Chapter 2 (page 39).

5.2.1.1 Notation

(1) $\mathcal{P}(X)$ is the power set of X, $\subseteq$ is the subset relation, $\subset$ the strict part of $\subseteq$, i.e. $A \subset B$ iff $A \subseteq B$ and $A \neq B$. The operators $\wedge$, $\neg$, $\vee$, $\rightarrow$ and $\vdash$ have their usual, classical interpretation.

(2) $\mathcal{I}(X) \subseteq \mathcal{P}(X)$ and $\mathcal{F}(X) \subseteq \mathcal{P}(X)$ are dual abstract notions of size, $\mathcal{I}(X)$ is the set of "small" subsets of X, $\mathcal{F}(X)$ the set of "big" subsets of X. They are dual in the sense that $A \in \mathcal{I}(X) \Leftrightarrow X - A \in \mathcal{F}(X)$. "$\mathcal{I}$" evokes "ideal", "$\mathcal{F}$" evokes "filter" though the full strength of both is reached only in $(< \omega * s)$. "s" evokes "small", and "$(x * s)$" stands for "x small sets together are still not everything".

(3) If $A \subseteq X$ is neither in $\mathcal{I}(X)$, nor in $\mathcal{F}(X)$, we say it has medium size, and we define $\mathcal{M}(X) := \mathcal{P}(X) - (\mathcal{I}(X) \cup \mathcal{F}(X))$. $\mathcal{M}^+(X) := \mathcal{P}(X) - \mathcal{I}(X)$ is the set of subsets which are not small.

(4) $\nabla x \phi$ is a generalized first order quantifier, it is read "almost all x have property ϕ". $\nabla x(\phi : \psi)$ is the relativized version, read: "almost all x with property ϕ have also property ψ". To keep the table "Rules on size" simple, we write mostly only the non-relativized versions.

Formally, we have $\nabla x\phi :\Leftrightarrow \{x : \phi(x)\} \in \mathcal{F}(U)$ where U is the universe, and $\nabla x(\phi : \psi) :\Leftrightarrow \{x : (\phi \wedge \psi)(x)\} \in \mathcal{F}(\{x : \phi(x)\})$. Soundness and completeness results on ∇ can be found in [Sch95-1].

(5) Analogously, for propositional logic, we define:
$\alpha \mid\!\sim \beta :\Leftrightarrow M(\alpha \wedge \beta) \in \mathcal{F}(M(\alpha))$,
where $M(\phi)$ is the set of models of ϕ.

(6) In preferential structures, $\mu(X) \subseteq X$ is the set of minimal elements of X. This generates a principal filter by $\mathcal{F}(X) := \{A \subseteq X : \mu(X) \subseteq A\}$. Corresponding properties about μ are not listed systematically.

(7) The usual rules (AND) etc. are named here (AND_ω), as they are in a natural ascending line of similar rules, based on strengthening of the filter/ideal properties.

(8) For any set of formulas T, and any consequence relation $\mid\!\sim$, we will use $\overline{T} := \{\phi : T \vdash \phi\}$, the set of classical consequences of T, and $\overline{\overline{T}} := \{\phi : T \mid\!\sim \phi\}$, the set of consequences of T under the relation $\mid\!\sim$.

(9) We say that a set X of models is definable by a formula (or a theory) iff there is a formula ϕ (a theory T) such that $X = M(\phi)$, or $X = M(T)$, the set of models of ϕ or T, respectively.

(10) Most rules are explained in the table "Logical rules", and "RW" stands for Right Weakening.

5.2.1.2 The groupes of rules

The rules concern properties of $\mathcal{I}(X)$ or $\mathcal{F}(X)$, or dependencies between such properties for different X and Y. All X, Y, etc. will be subsets of some universe, say V. Intuitively, V is the set of all models of some fixed propositional language. It is not necessary to consider all subsets of V, the intention is to consider subsets of V, which are definable by a formula or a theory. So we assume all X, Y etc. taken from some $\mathcal{Y} \subseteq \mathcal{P}(V)$, which we call the domain. In the former case, $\mathcal{Y}$ is closed under set difference, in the latter case not necessarily so. (We will mention it when we need some particular closure property.)

The rules are divided into 5 groups:

(1) (Opt), which says that "All" is optimal - i.e. when there are no exceptions, then a soft rule $\mid\!\sim$ holds.

(2) 3 monotony rules:

   (2.1) (iM) is inner monotony, a subset of a small set is small,

   (2.2) $(eM\mathcal{I})$ external monotony for ideals: enlarging the base set keeps small sets small,

   (2.3) $(eM\mathcal{F})$ external monotony for filters: a big subset stays big when the base set shrinks.

   These three rules are very natural if "size" is anything coherent over change of base sets. In particular, they can be seen as weakening.

(3) $(\approx)$ keeps proportions, it is here mainly to point the possibility out.

(4) a group of rules $x * s$, which say how many small sets will not yet add to the base set. The notation "$(< \omega * s)$" is an allusion to the full filter property, that filters are closed under $finite$ intersections.

(5) Rational monotony, which can best be understood as robustness of $\mathcal{M}^+$, see $(\mathcal{M}^{++})(3)$.

We will assume all base sets to be non-empty in order to avoid pathologies and in particular clashes between (Opt) and $(1 * s)$.

Note that the full strength of the usual definitions of a filter and an ideal are reached only in line $(< \omega * s)$.

Regularities

(1) The group of rules $(x * s)$ use ascending strength of $\mathcal{I}/\mathcal{F}$.

(2) The column $(\mathcal{M}^+)$ contains interesting algebraic properties. In particular, they show a strengthening from $(3 * s)$ up to Rationality. They are not necessarily equivalent to the corresponding (I_x) rules, not even in the presence of the basic rules. The examples show that care has to be taken when considering the different variants.

(3) Adding the somewhat superflous (CM_2), we have increasing cautious monotony from (wCM) to full (CM_ω).

(4) We have increasing "or" from (wOR) to full (OR_ω).

(5) The line $(2 * s)$ is only there because there seems to be no $(\mathcal{M}_2^+)$, otherwise we could begin $(n * s)$ at $n = 2$.

Summary

We can obtain all rules except $(RatM)$ and $(\approx)$ from (Opt), the monotony rules - (iM), $(eM\mathcal{I})$, $(eM\mathcal{F})$ -, and $(x * s)$ with increasing x.

5.2.1.3 Table

The following table is split in two, as it is too big for printing in one page.

(See Table 5.1 (page 189), "Rules on size - Part I" and Table 5.2 (page 190), "Rules on size - Part II".

5.2.2 Multiplicative laws about size

We are mainly interested in nonmonotonic logic. In this domain, independence is strongly connected to multiplication of abstract size, and an important part of the present text treats this connection and its repercussions.

We have at least two scenarios for multiplication, one is decribed in Diagram 5.2.1 (page 174), the second in Diagram 5.2.2 (page 176). In the first scenario, we have nested sets, in the second, we have set products. In the first scenario, we consider subsets which behave as the big set does, in the second scenario we consider subspaces, and decompose the behaviour of the big space into behaviour of the subspaces. In both cases, this results naturally in multiplication of abstract sizes. When we look at the corresponding relation properties, they are quite different (rankedness vs. some kind of modularity). But this is perhaps to be expected, as the two scenarios are quite different.

Other scenarios which might be interesting to consider in our framework are:

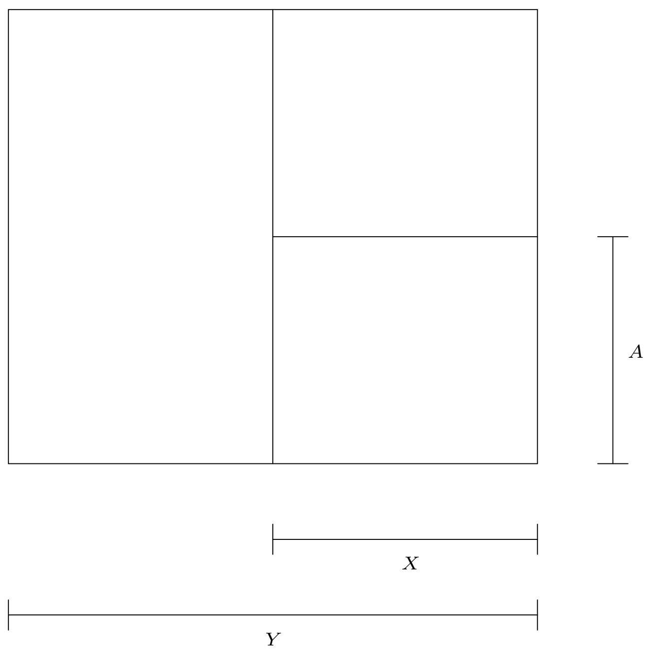
- When we have more than two truth values, say 3, and 2 is considered a big subset, and we have n propositional variables, and m of them are considered many, then 2^m might give a "big" subset of the total of $3^n situations$.
- Similarly, when we fix 1 variable, consider 2 cases of the possible 3, and multiply this with a "big" set of models.
- We may also consider the utility or cost of a situation, and work with a "big" utility, and "many" situations, etc.
- Note that, in the case of distances, subspaces add distances, and do not multiply them: $d(xy, x'y') = d(x, x') + d(y, y')$.

These questions are left for further research, see also Section 3.3 (page 104).

5.2.2.1 Multiplication of size for subsets

Here we have nested sets, $A \subseteq X \subseteq Y$, A is a certain proportion of X, and X of Y, resulting in a multiplication of relative size or proportions. This is a classical subject of nonmonotonic logic, see the last section, taken from [GS09a], it is partly repeated here to stress the common points with the other scenario.

Diagram 5.2.1



Scenario 1

Properties

Diagram 5.2.1 (page 174) is to be read as follows: The whole set Y is split in X and $Y - X$, X is split in A and $X - A$. X is a small/medium/big part of Y, A is a small/medium/big part of X. The question is: is A a small/medium/big part of Y?

Note that the relation of A to X is conceptually different from that of X to Y, as we change the base set by going from X to Y, but not when going from A to X. Thus, in particular, when we read the diagram as expressing multiplication, commutativity is not necessarily true.

We looked at this scenario already in [GS09a], but there from an additive point of view, using various basic properties like (iM), $(eM\mathcal{I})$, $(eM\mathcal{F})$, see Section 5.2.1 (page 171). Here, we use just multiplication - except sometimes for motivation.

We examine different rules:

If $Y = X$ or $X = A$, there is nothing to show, so 1 is the neutral element of multiplication.

If $X \in \mathcal{I}(Y)$ or $A \in \mathcal{I}(X)$, then we should have $A \in \mathcal{I}(Y)$. (Use for motivation (iM) or $(eM\mathcal{I})$ respectively.)

So it remains to look at the following cases, with the "natural" answers given already:

(1) $X \in \mathcal{F}(Y)$, $A \in \mathcal{F}(X) \Rightarrow A \in \mathcal{F}(Y)$,

(2) $X \in \mathcal{M}^+(Y)$, $A \in \mathcal{F}(X) \Rightarrow A \in \mathcal{M}^+(Y)$,

(3) $X \in \mathcal{F}(Y)$, $A \in \mathcal{M}^+(X) \Rightarrow A \in \mathcal{M}^+(Y)$,

(4) $X \in \mathcal{M}^+(Y)$, $A \in \mathcal{M}^+(X) \Rightarrow A \in \mathcal{M}^+(Y)$.

But (1) is case (3) of $(\mathcal{M}^+_\omega)$ in [GS09a], see Table "Rules on size" in Section 5.2.1 (page 171).

(2) is case (1) of $(\mathcal{M}^+_\omega)$ there,

(3) is case (2) of $(\mathcal{M}^+_\omega)$ there, finally,

(4) is $(\mathcal{M}^{++})$ there.

So the first three correspond to various expressions of (AND_ω), (OR_ω), (CM_ω), the last one to $(RatM)$.

But we can read them also the other way round, e.g.:

(1) corresponds to: $\alpha \mid\sim \beta$, $\alpha \wedge \beta \mid\sim \gamma \Rightarrow \alpha \mid\sim \gamma$,

(2) corresponds to: $\alpha \not\mid\sim \neg\beta$, $\alpha \wedge \beta \mid\sim \gamma \Rightarrow \alpha \not\mid\sim \neg(\beta \wedge \gamma)$,

(3) corresponds to: $\alpha \mid\sim \beta$, $\alpha \wedge \beta \not\mid\sim \neg\gamma \Rightarrow \alpha \not\mid\sim \neg(\beta \wedge \gamma)$.

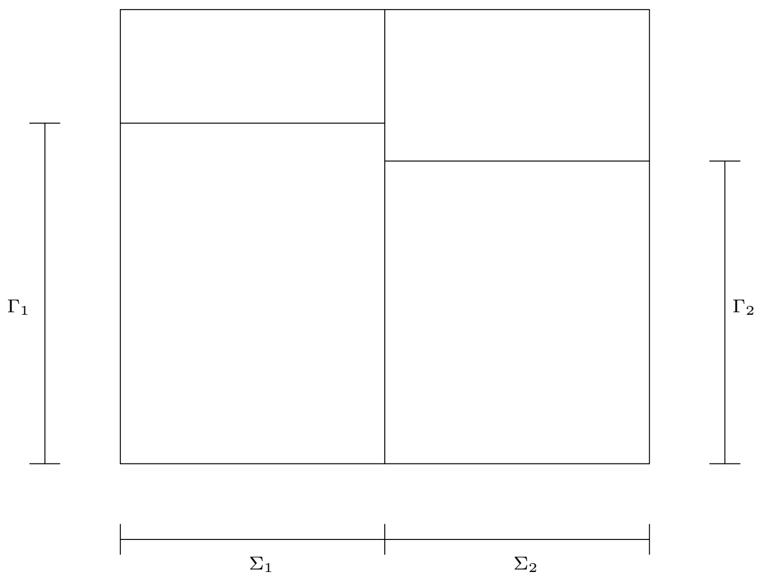
All these rules might be seen as too idealistic, so just as we did in [GS09a], we can consider milder versions: We might for instance consider a rule which says that $big * \ldots * big$, n times, is not small. Consider for instance the case $n = 2$. So we would conclude that A is not small in Y. In terms of logic, we then have: $\alpha \mid\sim \beta$, $\alpha \wedge \beta \mid\sim \gamma \Rightarrow \alpha \not\mid\sim (\neg\beta \vee \neg\gamma)$. We can obtain the same logical property from $3 * small \neq all$.

5.2.2.2 Multiplication of size for subspaces

Our main interest here is multiplication for subspaces, which we discuss now.

Properties

Diagram 5.2.2



Scenario 2

In this scenario, Σ_i are sets of sequences, see Diagram 5.2.2 (page 176), corresponding, intuitively, to a set of models in language $\mathcal{L}_i$, Σ_i will be the set of α_i-models, and the subsets Γ_i are to be seen as the "best" models, where β_i will hold. The languages are supposed to be disjoint sublanguages of a common language $\mathcal{L}$. As the Σ_i have symmetrical roles, there is no intuitive reason for multiplication not to be commutative.

We can interpret the situation twofold:

First, we work separately in sublanguage $\mathcal{L}_1$ and $\mathcal{L}_2$, and, say, α_i and β_i are both defined in $\mathcal{L}_i$, and we look at $\alpha_i \mid\sim \beta_i$ in the sublanguage $\mathcal{L}_i$, or, we consider both α_i and β_i in the big language $\mathcal{L}$, and look at $\alpha_i \mid\sim \beta_i$ in $\mathcal{L}$. These two ways are a priori completely different. Speaking in preferential terms, it is not at all clear why the orderings on the submodels should have anything to do with the orderings on the whole models. It seems a very desirable property, but we have to postulate it, which we do now (an overview is given in Table 5.3 (page 191)). We first give informally a list of such rules, mainly to show the connection with the first scenario. Later, see Definition 5.2.1 (page 178), we will introduce formally some rules for which we show a connection with interpolation. Here, e.g., "$(big * big \Rightarrow big)$" stands for "if both factors are big, so will be the product", this will be abbreviated by "$b * b \Rightarrow b$" in Table 5.3 (page 191) .

($big * 1 \Rightarrow big$) Let $\Gamma_1 \subseteq \Sigma_1$, if $\Gamma_1 \in \mathcal{F}(\Sigma_1)$, then $\Gamma_1 \times \Sigma_2 \in \mathcal{F}(\Sigma_1 \times \Sigma_2)$, (and the dual rule for Σ_2 and Γ_2).

This property preserves proportions, so it seems intuitively quite uncontested, whenever we admit coherence over products. (Recall that there was nothing to show in the first scenario.)

When we re-consider above case: suppose $\alpha \mid\!\sim \beta$ in the sublanguage, so $M(\beta) \in \mathcal{F}(M(\alpha))$ in the sublanguage, so by ($big * 1 \Rightarrow big$), $M(\beta) \in \mathcal{F}(M(\alpha))$ in the big language $\mathcal{L}$.

We obtain the dual rule for small (and likewise, medium size) sets:

($small * 1 \Rightarrow small$) Let $\Gamma_1 \subseteq \Sigma_1$, if $\Gamma_1 \in \mathcal{I}(\Sigma_1)$, then $\Gamma_1 \times \Sigma_2 \in \mathcal{I}(\Sigma_1 \times \Sigma_2)$, (and the dual rule for Σ_2 and Γ_2),

establishing $All = 1$ as the neutral element for multiplication.

We look now at other, plausible rules:

($small * x \Rightarrow small$) $\Gamma_1 \in \mathcal{I}(\Sigma_1)$, $\Gamma_2 \subseteq \Sigma_2 \Rightarrow \Gamma_1 \times \Gamma_2 \in \mathcal{I}(\Sigma_1 \times \Sigma_2)$

($big * big \Rightarrow big$) $\Gamma_1 \in \mathcal{F}(\Sigma_1)$, $\Gamma_2 \in \mathcal{F}(\Sigma_2) \Rightarrow \Gamma_1 \times \Gamma_2 \in \mathcal{F}(\Sigma_1 \times \Sigma_2)$

($big * medium \Rightarrow medium$) $\Gamma_1 \in \mathcal{F}(\Sigma_1)$, $\Gamma_2 \in \mathcal{M}^+(\Sigma_2) \Rightarrow \Gamma_1 \times \Gamma_2 \in \mathcal{M}^+(\Sigma_1 \times \Sigma_2)$

($medium * medium \Rightarrow medium$) $\Gamma_1 \in \mathcal{M}^+(\Sigma_1)$, $\Gamma_2 \in \mathcal{M}^+(\Sigma_2) \Rightarrow \Gamma_1 \times \Gamma_2 \in \mathcal{M}^+(\Sigma_1 \times \Sigma_2)$

When we accept all above rules, we can invert ($big * big \Rightarrow big$), as a big product must be composed of big components. Likewise, at least one component of a small product has to be small - see Proposition 5.2.1 (page 179).

We see that these properties give a lot of modularity. We can calculate the consequences of α and α' separately - provided α, α' use disjoint alphabets - and put the results together afterwards. Such properties are particularly interesting for classification purposes, where subclasses are defined with disjoint alphabets.

Recall that we work here with a notion of “big” and “small” subsets, which may be thought of as defined by a filter (ideal), though we usually will not need the full strength of a filter (ideal). But assume as usual that $A \subseteq B \subseteq C$ and $A \subseteq C$ is big imply $B \subseteq C$ is big, that $C \subseteq C$ is big, and define $A \subseteq B$ is small iff $(B - A) \subseteq B$ is big, call all subsets which are neither big nor small medium size. For an extensive discussion, see [GS09a].

Let $X' \cup X'' = X$ be a disjoint cover, so $\Pi X = \Pi X' \times \Pi X''$. We consider subsets Σ etc. of ΠX. If not said otherwise, Σ etc. need not be a product $\Sigma' \times \Sigma''$. We will sometimes write $\Pi' := \Pi X'$, $\Sigma'' := \Sigma \restriction X''$. The roles of X' and X'' are interchangeable, e.g., instead of $\Gamma \restriction X' \subseteq \Sigma \restriction X'$, we may also write $\Gamma \restriction X'' \subseteq \Sigma \restriction X''$.

We consider here the following two sets of three finite product rules about size and μ. Both sets will be shown to be equivalent in Proposition 5.2.1 (page 179).

Definition 5.2.1

$(S * 1)$ $\Delta \subseteq \Sigma' \times \Sigma''$ is big iff there is $\Gamma = \Gamma' \times \Gamma'' \subseteq \Delta$ such that $\Gamma' \subseteq \Sigma'$ and $\Gamma'' \subseteq \Sigma''$ are big

$(S * 2)$ $\Gamma \subseteq \Sigma$ is big $\Rightarrow \Gamma \restriction X' \subseteq \Sigma \restriction X'$ is big - where Σ is not necessarily a product.

$(S * 3)$ $A \subseteq \Sigma$ is big $\Rightarrow$ there is $B \subseteq \Pi' \times \Sigma''$ big such that $B \restriction X'' \subseteq A \restriction X''$ - again, Σ is not necessarily a product.

$(\mu * 1)$ $\mu(\Sigma' \times \Sigma'') = \mu(\Sigma') \times \mu(\Sigma'')$

$(\mu * 2)$ $\mu(\Sigma) \subseteq \Gamma \Rightarrow \mu(\Sigma \restriction X') \subseteq \Gamma \restriction X'$

$(\mu * 3)$ $\mu(\Pi X' \times \Sigma'') \restriction X'' \subseteq \mu(\Sigma) \restriction X''$

$(s * s)$ Let $\Gamma_i \subseteq \Sigma_i$, then $\Gamma_1 \times \Gamma_2 \subseteq \Sigma_1 \times \Sigma_2$ is small iff $\Gamma_1 \subseteq \Sigma_1$ is small or $\Gamma_1 \subseteq \Sigma_1$ is small.

A generalization to more than two factors is obvious.

One can also consider weakenings, e.g.,

$(S * 1')$ $\Gamma' \times \Sigma'' \subseteq \Sigma' \times \Sigma''$ is big iff $\Gamma' \subseteq \Sigma'$ is big.

Proposition 5.2.1

(1) Let $(S * 1)$ hold. Then:
$\Gamma' \times \Gamma'' \subseteq \Sigma' \times \Sigma''$ is small iff $\Gamma' \subseteq \Sigma'$ or $\Gamma'' \subseteq \Sigma''$ is small.

(2) If the filters over A are principal filters, generated by $\mu(A)$, i.e. $B \subseteq A$ is big iff $\mu(A) \subseteq B \subseteq A$ for some $\mu(A) \subseteq A$, then:
$(S * i)$ is equivalent to $(\mu * i)$, $i = 1, 2, 3$.

(3) Let the notion of size satisfy (Opt), (iM), and $(< \omega * s)$, see the tables "Rules on size" in Section 5.2.1 (page 171). Then $(\mu * 1)$ and $(s * s)$ are equivalent.

Proof

(1)

"$\Leftarrow$":

Suppose $\Gamma' \subseteq \Sigma'$ is small. Then $\Sigma' - \Gamma' \subseteq \Sigma'$ is big and $(\Sigma' - \Gamma') \times \Sigma'' \subseteq \Sigma' \times \Sigma''$ is big by $(S * 1)$. But $(\Gamma' \times \Gamma'') \cap ((\Sigma' - \Gamma') \times \Sigma'') = \emptyset$, so $\Gamma' \times \Gamma'' \subseteq \Sigma' \times \Sigma''$ is small.

"$\Rightarrow$":

For the converse, suppose that neither $\Gamma' \subseteq \Sigma'$ nor $\Gamma'' \subseteq \Sigma''$ are small. Let $A \subseteq \Sigma' \times \Sigma''$ be big, we show that $A \cap (\Gamma' \times \Gamma'') \neq \emptyset$. By $(S * 1)$ there are $B' \subseteq \Sigma'$ and $B'' \subseteq \Sigma''$ big, and $B' \times B'' \subseteq A$. Then $B' \cap \Gamma' \neq \emptyset$, $B'' \cap \Gamma'' \neq \emptyset$, so there is $\langle x', x'' \rangle \in (B' \times B'') \cap (\Gamma' \times \Gamma'') \subseteq A \cap (\Gamma' \times \Gamma'')$.

(2.1)

"$\Rightarrow$"

"$\subseteq$": $\mu(\Sigma') \subseteq \Sigma'$ and $\mu(\Sigma'') \subseteq \Sigma''$ are big, so by $(S * 1)$ $\mu(\Sigma') \times \mu(\Sigma'') \subseteq \Sigma' \times \Sigma''$ is big, so $\mu(\Sigma' \times \Sigma'') \subseteq \mu(\Sigma') \times \mu(\Sigma'')$.

"$\supseteq$": $\mu(\Sigma' \times \Sigma'') \subseteq \Sigma' \times \Sigma''$ is big $\Rightarrow$ by $(S * 1)$ there is $\Gamma' \times \Gamma'' \subseteq \mu(\Sigma' \times \Sigma'')$ and $\Gamma' \subseteq \Sigma'$, $\Gamma'' \subseteq \Sigma''$ big $\Rightarrow \mu(\Sigma') \subseteq \Gamma'$, $\mu(\Sigma'') \subseteq \Gamma'' \Rightarrow \mu(\Sigma') \times \mu(\Sigma'') \subseteq \mu(\Sigma' \times \Sigma'')$.

"$\Leftarrow$"

Let $\Gamma' \subseteq \Sigma'$ be big, $\Gamma'' \subseteq \Sigma''$ be big, $\Gamma' \times \Gamma'' \subseteq \Delta$, then $\mu(\Sigma') \subseteq \Gamma'$, $\mu(\Sigma'') \subseteq \Gamma''$, so by $(\mu * 1)$ $\mu(\Sigma) = \mu(\Sigma') \times \mu(\Sigma'') \subseteq \Gamma' \times \Gamma'' \subseteq \Delta$, so Δ is big.

Let $\Delta \subseteq \Sigma$ be big, then by $(\mu * 1)$ $\mu(\Sigma') \times \mu(\Sigma'') = \mu(\Sigma) \subseteq \Delta$.

(2.2)

"$\Rightarrow$"

$\mu(\Sigma) \subseteq \Gamma \Rightarrow \Gamma \subseteq \Sigma$ big $\Rightarrow$ by $(S*2)$ $\Gamma \restriction X' \subseteq \Sigma \restriction X'$ big $\Rightarrow \mu(\Sigma \restriction X') \subseteq \Gamma \restriction X'$.

"$\Leftarrow$"

$\Gamma \subseteq \Sigma$ big $\Rightarrow \mu(\Sigma) \subseteq \Gamma \Rightarrow$ by $(\mu*2)$ $\mu(\Sigma \restriction X') \subseteq \Gamma \restriction X' \Rightarrow \Gamma \restriction X' \subseteq \Sigma \restriction X'$ big.

(2.3)

"$\Rightarrow$"

$\mu(\Sigma) \subseteq \Sigma$ big $\Rightarrow \exists B \subseteq \Pi X' \times \Sigma''$ big such that $B \restriction X'' \subseteq \mu(\Sigma) \restriction X''$ by $(S*3)$, thus in particular $\mu(\Pi X' \times \Sigma'') \restriction X'' \subseteq \mu(\Sigma) \restriction X''$.

"$\Leftarrow$"

$A \subseteq \Sigma$ big $\Rightarrow \mu(\Sigma) \subseteq A$. $\mu(\Pi X' \times \Sigma'') \subseteq \Pi X' \times \Sigma''$ is big, and by $(\mu*3)$ $\mu(\Pi X' \times \Sigma'') \restriction X'' \subseteq \mu(\Sigma) \restriction X'' \subseteq A \restriction X''$.

(3)

"$\Rightarrow$":

(1) Let $\Gamma' \subseteq \Sigma'$ be small, we show that $\Gamma' \times \Gamma'' \subseteq \Sigma' \times \Sigma''$ is small. So $\Sigma' - \Gamma' \subseteq \Sigma'$ is big, so by (Opt) and $(\mu*1)$ $(\Sigma' - \Gamma') \times \Sigma'' \subseteq \Sigma' \times \Sigma''$ is big, so $\Gamma' \times \Sigma'' = (\Sigma' \times \Sigma'') - ((\Sigma' - \Gamma') \times \Sigma'') \subseteq \Sigma' \times \Sigma''$ is small, so by (iM) $\Gamma' \times \Gamma'' \subseteq \Sigma' \times \Sigma''$ is small.

(2) Suppose $\Gamma' \subseteq \Sigma'$ and $\Gamma'' \subseteq \Sigma''$ are not small, we show that $\Gamma' \times \Gamma'' \subseteq \Sigma' \times \Sigma''$ is not small. So $\Sigma' - \Gamma' \subseteq \Sigma'$ and $\Sigma'' - \Gamma'' \subseteq \Sigma''$ are not big. We show that $Z := ((\Sigma' \times \Sigma'') - (\Gamma' \times \Gamma'')) \subseteq \Sigma' \times \Sigma''$ is not big. $Z = (\Sigma' \times (\Sigma'' - \Gamma'')) \cup ((\Sigma' - \Gamma') \times \Sigma'')$.

Suppose $X' \times X'' \subseteq Z$, then $X' \subseteq \Sigma' - \Gamma'$ or $X'' \subseteq \Sigma'' - \Gamma''$. Proof: Let $X' \not\subseteq \Sigma' - \Gamma'$ and $X'' \not\subseteq \Sigma'' - \Gamma''$, but $X' \times X'' \subseteq Z$. Let $\sigma' \in X' - (\Sigma' - \Gamma')$, $\sigma'' \in X'' - (\Sigma'' - \Gamma'')$, consider $\sigma'\sigma''$. $\sigma'\sigma'' \notin (\Sigma' - \Gamma') \times \Sigma''$, as $\sigma' \notin \Sigma' - \Gamma'$, $\sigma'\sigma'' \notin \Sigma' \times (\Sigma'' \times \Gamma'')$, as $\sigma'' \notin \Sigma'' - \Gamma''$, so $\sigma'\sigma'' \notin Z$.

By prerequisite, $\Sigma' - \Gamma' \subseteq \Sigma'$ is not big, $\Sigma'' - \Gamma'' \subseteq \Sigma''$ is not big, so by (iM) no X' with $X' \subseteq \Sigma' - \Gamma'$ is big, no X'' with $X'' \subseteq \Sigma'' - \Gamma''$ is big, so by $(\mu*1)$ or $(S*1)$ $Z \subseteq \Sigma' \times \Sigma''$ is not big, so $\Gamma' \times \Gamma'' \subseteq \Sigma' \times \Sigma''$ is not small.

"$\Leftarrow$":

(1) Suppose $\Gamma' \subseteq \Sigma'$ is big, $\Gamma'' \subseteq \Sigma''$ is big, we have to show $\Gamma' \times \Gamma'' \subseteq \Sigma' \times \Sigma''$ is big. $\Sigma' - \Gamma' \subseteq \Sigma'$ is small, $\Sigma'' - \Gamma'' \subseteq \Sigma''$ is small, so by $(s*s)$ $(\Sigma' - \Gamma') \times \Sigma'' \subseteq \Sigma' \times \Sigma''$ is small and $\Sigma' \times (\Sigma'' - \Gamma'') \subseteq \Sigma' \times \Sigma''$ is small, so by $(< \omega * s)$ $(\Sigma' \times \Sigma'') - (\Gamma' \times \Gamma'') = ((\Sigma' - \Gamma') \times \Sigma'') \cup (\Sigma' \times (\Sigma'' - \Gamma'')) \subseteq \Sigma' \times \Sigma''$ is small, so $\Gamma' \times \Gamma'' \subseteq \Sigma' \times \Sigma''$ is big.

(2) Suppose $\Gamma' \times \Gamma'' \subseteq \Sigma' \times \Sigma''$ is big, we have to show $\Gamma' \subseteq \Sigma'$ is big, and $\Gamma'' \subseteq \Sigma''$ is big. By prerequisite, $(\Sigma' \times \Sigma'') - (\Gamma' \times \Gamma'') = ((\Sigma' - \Gamma') \times \Sigma'') \cup (\Sigma' \times (\Sigma'' - \Gamma'')) \subseteq \Sigma' \times \Sigma''$ is small, so by (iM) $\Sigma' \times (\Sigma'' - \Gamma'') \subseteq \Sigma' \times \Sigma''$ is small, so by (Opt) and $(s*s)$ $\Sigma'' - \Gamma'' \subseteq \Sigma''$ is small, so $\Gamma'' \subseteq \Sigma''$ is big, and likewise $\Gamma' \subseteq \Sigma'$ is big.

□

Discussion

We compare these rules to probability defined size.

Let "big" be defined by "more than 50%". If $\Pi X'$ and $\Pi X''$ have 3 elements each, then subsets of $\Pi X'$ or $\Pi X''$ of *card* ≥ 2 are big. But taking the product may give $4/9 < 1/2$. So the product rule "$big * big = big$" will not hold there. One direction will hold, of course.

Next, we discuss the prerequisite $\Sigma = \Sigma' \times \Sigma''$. Consider the following example:

Example 5.2.1

Take a language of 5 propositional variables, with $X' := \{a, b, c\}$, $X'' := \{d, e\}$. Consider the model set $\Sigma := \{\pm a \pm b \pm cde, -a-b-c-d\pm e\}$, i.e. of 8 models of de and 2 models of $-d$. The models of de are 8/10 of all elements of Σ, so it is reasonable to call them a big subset of Σ. But its projection on X'' is only 1/3 of Σ''.

So we have a potential *decrease* when going to the coordinates.

This shows that weakening the prerequisite about Σ as done in $(S * 2)$ is not innocent.

Remark 5.2.2

When we set small sets to 0, big sets to 1, we have the following boolean rules for filters:

(1) $0 + 0 = 0$

(2) $1 + x = 1$

(3) $-0 = 1$, $-1 = 0$

(4) $0 * x = 0$

(5) $1 * 1 = 1$

There are no such rules for medium size sets, as the union of two medium size sets may be big, but also stay medium.

Such multiplication rules capture the behaviour of Reiter defaults and of defeasible inheritance.

5.2.3 Hamming relations and distances

5.2.3.1 Hamming relations and multiplication of size

We now define Hamming relations in various flavours, and then (see Proposition 5.2.4 (page 182)) show that (smooth) Hamming relations generate a notion of size which satisfies our conditions, defined in Definition 5.2.1 (page 178). Corollary 5.3.4 (page 198) will put our results together, and show that (smooth) Hamming relations generate preferential logics with interpolation.

We will conclude this section by showing that our conditions $(\mu * 1)$ and $(\mu * 2)$ essentially characterise Hamming relations.

Note that we re-define Hamming relations in Section 6.3.1.3 (page 220), as already announced in Section 1.5.4.4 (page 29).

Definition 5.2.2

We abuse notation, and define a relation $\subseteq$ on $\Pi X'$, $\Pi X''$, and $\Pi X' \times \Pi X''$.

(1) Define $x \prec y :\Leftrightarrow x \preceq y$ and $x \neq y$, thus $\sigma \preceq \tau$ iff $\sigma \prec \tau$ or $\sigma = \tau$.

(2) We say that a relation $\preceq$ satisfies (GH3) iff

(GH3) $\sigma'\sigma'' \preceq \tau'\tau'' \Leftrightarrow \sigma' \preceq \tau'$ and $\sigma'' \preceq \tau''$, (Thus, $\sigma'\sigma'' \prec \tau'\tau''$ iff $\sigma'\sigma'' \preceq \tau'\tau''$ and ($\sigma' \prec \tau'$ or $\sigma'' \prec \tau''$).)

(3) Call a relation $\prec$ a GH (= general Hamming) relation iff the following two conditions hold:

$(GH1)$ $\sigma' \preceq \tau' \wedge \sigma'' \preceq \tau'' \wedge (\sigma' \prec \tau' \vee \sigma'' \prec \tau'') \Rightarrow \sigma'\sigma'' \prec \tau'\tau''$

(where $\sigma' \preceq \tau'$ iff $\sigma' \prec \tau'$ or $\sigma' = \tau'$)

$(GH2)$ $\sigma'\sigma'' \prec \tau'\tau'' \Rightarrow \sigma' \prec \tau' \vee \sigma'' \prec \tau''$

$(GH2)$ means that some compensation is possible, e.g., $\tau' \prec \sigma'$ might be the case, but $\sigma'' \prec \tau''$ wins in the end, so $\sigma'\sigma'' \prec \tau'\tau''$.

We use (GH) for $(GH1) + (GH2)$.

Example 5.2.2

The circumscription relation satisfies (GH3) with $\neg p \leq p$ and $\bigwedge \pm q_i \leq \bigwedge \pm q'_i$ iff $\forall i(\pm q_i \leq \pm q'_i)$.

Remark 5.2.3

(1) The independence makes sense because the concept of models, and thus the usual interpolation for classical logic relie on the independence of the assignments.

(2) This corresponds to social choice for many independent dimensions.

(3) We can also consider such factorisation as an approximation: we can do part of the reasoning independently.

Definition 5.2.3

Given a relation $\preceq$, define as usual a principal filter $\mathcal{F}(X)$ generated by the $\preceq$ −minimal elements:

$\mu(X) := \{x \in X : \neg\exists x' \prec x.x' \in X\}$,

$\mathcal{F}(X) := \{A \subseteq X : \mu(X) \subseteq A\}$.

The following proposition summarizes various properties for the different Hamming relations:

Proposition 5.2.4

Let $\Pi X = \Pi X' \times \Pi X''$, $\Sigma \subseteq \Pi X$.

(1) Let $\preceq$ be a smooth relation satisfying (GH3). Then $(\mu{*}2)$ holds, and thus $(S{*}2)$ by Proposition 5.2.1 (page 179), (2).

(2) Let again $\Sigma' := \Sigma \restriction X'$, $\Sigma'' := \Sigma \restriction X''$. Let $\preceq$ be a smooth relation satisfying (GH3). Then:

$\mu(\Sigma') \times \mu(\Sigma'') \subseteq \Sigma \Rightarrow \mu(\Sigma) = \mu(\Sigma') \times \mu(\Sigma'')$.

(Here $\Sigma = \Sigma' \times \Sigma''$ will not necessarily hold.)

(3) Let again $\Sigma' := \Sigma \restriction X'$, $\Sigma'' := \Sigma \restriction X''$. Let $\preceq$ be a relation satisfying (GH3), and $\Sigma = \Sigma' \times \Sigma''$. Then $(\mu * 1)$ holds, and thus, by Proposition 5.2.1 (page 179), (2) $(S * 1)$.

(4) Let $\preceq$ be a smooth relation satisfying (GH3), then $(\mu * 3)$ holds, and thus by Proposition 5.2.1 (page 179) (2) $(S * 3)$.

(5)

$(\mu{*}1)$ and $(\mu{*}2)$ and the usual axioms for smooth relations characterize smooth relations satisfying (GH3).

(6)

Let $\sigma \prec \tau \Leftrightarrow \tau \notin \mu(\{\sigma, \tau\})$ and $\prec$ be smooth. Then μ satisfies $(\mu * 1)$ (or, by Proposition 5.2.1 (page 179) equivalently $(s * s)$) iff $\prec$ is a GH relation.

(7) Let $\Gamma' \subseteq \Sigma'$, $\Gamma'' \subseteq \Sigma''$, $\Gamma' \times \Gamma'' \subseteq \Sigma' \times \Sigma''$ be small, let $(GH2)$ hold, then $\Gamma' \subseteq \Sigma'$ is small or $\Gamma'' \subseteq \Sigma''$ is small.

(8) Let $\Gamma' \subseteq \Sigma'$ be small, $\Gamma'' \subseteq \Sigma''$, let $(GH1)$ hold, then $\Gamma' \times \Gamma'' \subseteq \Sigma' \times \Sigma''$ is small.

Proof

(1) Suppose $\mu(\Sigma) \subseteq \Gamma$ and $\sigma' \in \Sigma \upharpoonright X' - \Gamma \upharpoonright X'$, we show $\sigma' \notin \mu(\Sigma \upharpoonright X')$.

Let $\sigma = \sigma'\sigma'' \in \Sigma$, then $\sigma \notin \Gamma$, so $\sigma \notin \mu(\Sigma)$. So here is $\rho \prec \sigma$, $\rho \in \mu(\Sigma) \subseteq \Gamma$ by smoothness. Let $\rho = \rho'\rho''$. We have $\rho' \preceq \sigma'$ by (GH3). $\rho' = \sigma'$ cannot be, as $\rho' \in \Gamma \upharpoonright X'$, and $\sigma' \notin \Gamma \upharpoonright X'$. So $\rho' \prec \sigma'$, and $\sigma' \notin \mu(\Sigma \upharpoonright X')$.

(2)

"$\supseteq$": Let $\sigma' \in \mu(\Sigma')$, $\sigma'' \in \mu(\Sigma'')$. By prerequisite, $\sigma'\sigma'' \in \Sigma$. Suppose $\tau \prec \sigma'\sigma''$, then $\tau' \prec \sigma'$ or $\tau'' \prec \sigma''$, contradiction.

"$\subseteq$": Let $\sigma \in \mu(\Sigma)$, suppose $\sigma' \notin \mu(\Sigma')$ or $\sigma'' \notin \mu(\Sigma'')$. So there are $\tau' \preceq \sigma'$, $\tau'' \preceq \sigma''$ with $\tau' \in \mu(\Sigma')$, $\tau'' \in \mu(\Sigma'')$ by smoothness. Moreoever, $\tau' \prec \sigma'$ or $\tau'' \prec \sigma''$. By prerequisite $\tau'\tau'' \in \Sigma$, and $\tau'\tau'' \prec \sigma$, so $\sigma \notin \mu(\Sigma)$.

(3)

"$\supseteq$": As in (2), the prerequisite holds trivially.

"$\subseteq$": As in (2), but we do not need $\tau' \in \mu(\Sigma')$, $\tau'' \in \mu(\Sigma'')$, as $\tau'\tau''$ will be in Σ trivially. So smoothness is not needed.

(4)

Let again $\Sigma'' = \Sigma \upharpoonright X''$.

Let $\Delta := \Pi X' \times \Sigma''$, $\sigma = \sigma'\sigma'' \in \mu(\Delta)$. Suppose $\sigma'' \notin \mu(\Sigma) \upharpoonright X''$. There cannot be any $\tau \prec \sigma$, $\tau \in \Sigma$, by $\Sigma \subseteq \Delta$. So $\sigma \notin \Sigma$, but $\sigma'' \in \Sigma''$, so there is $\tau \in \Sigma$ $\tau'' = \sigma''$. As τ is not minimal, there must be minimal $\rho = \rho'\rho'' \prec \tau$, $\rho \in \Sigma$ by smoothness. As ρ is minimal, $\rho'' \neq \sigma''$, and as $\rho \prec \tau$, $\rho'' \prec \sigma''$ by (GH3). By prerequisite $\sigma'\rho'' \in \Delta$, and $\sigma'\rho'' \prec \sigma$, contradiction.

Note that smoothness is essential. Otherwise, there might be an infinite descending chain τ_i below τ, all with $\tau_i'' = \sigma''$, but none below σ.

(5)

If $\preceq$ is smooth and satisfies (GH3), then $(\mu * 1)$ and $(\mu * 2)$ hold by (1) and (3). For the converse:

Define as usual $\sigma \prec \tau :\Leftrightarrow \tau \notin \mu(\{\sigma, \tau\})$. Let $\sigma = \sigma'\sigma''$, $\tau = \tau'\tau''$.

We have to show:

$\sigma \prec \tau$ iff $\sigma' \preceq \tau'$ and $\sigma'' \preceq \tau''$ and ($\sigma' \prec \tau'$ or $\sigma'' \prec \tau''$).

"$\Leftarrow$":

Suppose $\sigma' \prec \tau'$ and $\sigma'' \preceq \tau''$. Then $\mu(\{\sigma', \tau'\}) = \{\sigma'\}$, and $\mu(\{\sigma'', \tau''\}) = \{\sigma''\}$ (either $\sigma'' \prec \tau''$ or $\sigma'' = \tau''$, so in both cases $\mu(\{\sigma'', \tau''\}) = \{\sigma''\}$). As $\tau' \notin \mu(\{\sigma', \tau'\})$, $\tau \notin \mu(\{\sigma', \tau'\} \times \{\sigma'', \tau''\})$ = (by $(\mu * 1)$) $\mu(\{\sigma', \tau'\}) \times \mu(\{\sigma'', \tau''\}) = \{\sigma'\} \times \{\sigma''\} = \{\sigma\}$, so by smoothness $\sigma \prec \tau$.

"$\Rightarrow$":

Conversely, if $\sigma \prec \tau$, so $\Gamma := \{\sigma\} = \mu(\Sigma)$ for $\Sigma := \{\sigma, \tau\}$, so by $(\mu * 2)$ $\mu(\Sigma \restriction X') = \mu(\{\sigma', \tau'\}) \subseteq \Gamma \restriction X' = \{\sigma'\}$, so $\sigma' \preceq \tau'$, analogously $\mu(\Sigma \restriction X'') = \mu(\{\sigma'', \tau''\}) \subseteq \Gamma \restriction X'' = \{\sigma''\}$, so $\sigma'' \preceq \tau''$, but both cannot be equal.

(6)

(6.1) $(\mu * 1)$ entails the GH relation conditions

$(GH1)$: Suppose $\sigma' \prec \tau'$ and $\sigma'' \preceq \tau''$. Then $\tau' \notin \mu(\{\sigma', \tau'\}) = \{\sigma'\}$, and $\mu(\{\sigma'', \tau''\}) = \{\sigma''\}$ (either $\sigma'' \prec \tau''$ or $\sigma'' = \tau''$, so in both cases $\mu(\{\sigma'', \tau''\}) = \{\sigma''\}$). As $\tau' \notin \mu(\{\sigma', \tau'\})$, $\tau'\tau'' \notin \mu(\{\sigma', \tau'\} \times \{\sigma'', \tau''\}) =_{(\mu*1)} \mu(\{\sigma', \tau'\}) \times \mu(\{\sigma'', \tau''\}) = \{\sigma'\} \times \{\sigma''\} = \{\sigma'\sigma''\}$, so by smoothness $\sigma'\sigma'' \prec \tau'\tau''$.

$(GH2)$: Let $X := \{\sigma', \tau'\}$, $Y := \{\sigma'', \tau''\}$, so $X \times Y = \{\sigma'\sigma'', \sigma'\tau'', \tau'\sigma'', \tau'\tau''\}$. Suppose $\sigma'\sigma'' \prec \tau'\tau''$, so $\tau'\tau'' \notin \mu(X \times Y) =_{(\mu*1)} \mu(X) \times \mu(Y)$. If $\sigma' \not\prec \tau'$, then $\tau' \in \mu(X)$, likewise if $\sigma'' \not\prec \tau''$, then $\tau'' \in \mu(Y)$, so $\tau'\tau'' \in \mu(X \times Y)$, contradiction.

(6.2) The GH relation conditions generate $(\mu * 1)$.

$\mu(X \times Y) \subseteq \mu(X) \times \mu(Y)$: Let $\tau' \in X$, $\tau'' \in Y$, $\tau'\tau'' \notin \mu(X) \times \mu(Y)$, then $\tau' \notin \mu(X)$ or $\tau'' \notin \mu(Y)$. Suppose $\tau' \notin \mu(X)$, let $\sigma' \in X$, $\sigma' \prec \tau'$, so by condition $(GH1)$ $\sigma'\tau'' \prec \tau'\tau''$, so $\tau'\tau'' \notin \mu(X \times Y)$.

$\mu(X) \times \mu(Y) \subseteq \mu(X \times Y)$: Let $\tau' \in X$, $\tau'' \in Y$, $\tau'\tau'' \notin \mu(X \times Y)$, so there is $\sigma'\sigma'' \prec \tau'\tau''$, $\sigma' \in X$, $\sigma'' \in Y$, so by $(GH2)$ either $\sigma' \prec \tau'$ or $\sigma'' \prec \tau''$, so $\tau' \notin \mu(X)$ or $\tau'' \notin \mu(Y)$, so $\tau'\tau'' \notin \mu(X) \times \mu(Y)$.

(7)

Suppose $\Gamma' \subseteq \Sigma'$ is not small, so there is $\gamma' \in \Gamma'$ and no $\sigma' \in \Sigma'$ with $\sigma' \prec \gamma'$. Fix this γ'. Consider $\{\gamma'\} \times \Gamma''$. As $\Gamma' \times \Gamma'' \subseteq \Sigma' \times \Sigma''$ is small, there is for each $\gamma'\gamma''$, $\gamma'' \in \Gamma''$ some $\sigma'\sigma'' \in \Sigma' \times \Sigma''$, $\sigma'\sigma'' \prec \gamma'\gamma''$. By $(GH2)$ $\sigma' \prec \gamma'$ or $\sigma'' \prec \gamma''$, but $\sigma' \prec \gamma'$ was excluded, so for all $\gamma'' \in \Gamma''$ there is $\sigma'' \in \Sigma''$ with $\sigma'' \prec \gamma''$, so $\Gamma'' \subseteq \Sigma''$ is small.

(8)

Let $\gamma' \in \Gamma'$, so there is $\sigma' \in \Sigma'$ and $\sigma' \prec \gamma'$. By $(GH1)$, for any $\gamma'' \in \Gamma''$ $\sigma'\gamma'' \prec \gamma'\gamma''$, so no $\gamma'\gamma'' \in \Gamma' \times \Gamma''$ is minimal.

$\square$

Example 5.2.3

Even for smooth relations satisfying (GH3), the converse of $(\mu * 2)$ is not necessarily true:

Let $\sigma' \prec \tau'$, $\tau'' \prec \sigma''$, $\Sigma := \{\sigma, \tau\}$, then $\mu(\Sigma) = \Sigma$, but $\mu(\Sigma') = \{\sigma'\}$, $\mu(\Sigma'') = \{\tau''\}$, so $\mu(\Sigma) \neq \mu(\Sigma') \times \mu(\Sigma'')$.

We need the additional assumption that $\mu(\Sigma') \times \mu(\Sigma'') \subseteq \Sigma$, see Proposition 5.2.4 (page 182) (2).

Example 5.2.4

The following are examples of GH relations:

Define on all components X_i a relation $\prec_i$.

(1) The set variant Hamming relation:

Let the relation $\prec$ be defined on $\Pi\{X_i : i \in I\}$ by $\sigma \prec \tau$ iff for all j $\sigma_j \preceq_j \tau_j$, and there is at least one i such that $\sigma_i \prec_i \tau_i$.

(2) The counting variant Hamming relation:

Let the relation $\prec$ be defined on $\Pi\{X_i : i \in I\}$ by $\sigma \prec \tau$ iff the number of i such that $\sigma_i \prec_i \tau_i$ is bigger than the number of i such that $\tau_i \prec_i \sigma_i$.

(3) The weighed counting Hamming relation:

Like the counting relation, but we give different (numerical) importance to different i. E.g., $\sigma_1 \prec \tau_1$ may count 1, $\sigma_2 \prec \tau_2$ may count 2, etc.

□

Note

Note that already $(\mu * 1)$ results in a strong independence result in the second scenario: Let $\sigma\rho' \prec \tau\rho'$, then $\sigma\rho'' \prec \tau\rho''$ for all ρ''. Thus, whether $\{\rho''\}$ is small, or medium size (i.e. $\rho'' \in \mu(\Sigma')$), the behaviour of $\Sigma \times \{\rho''\}$ is the same. This we do not have in the first scenario, as small sets may behave very differently from medium size sets. (But, still, their internal structure is the same, only the minimal elements change.) When $(\mu * 2)$ holds, then if $\sigma\sigma' \prec \tau\tau'$ and $\sigma \neq \tau$, then $\sigma \prec \tau$, i.e. we need not have $\sigma' = \tau'$.

5.2.3.2 Hamming distances and revision

This short Section is mainly intended to put our work in a broader perspective, by showing a connection of Hamming distances to modular revision as introduced by Parikh and his co-authors. The main result here is Corollary 5.2.7 (page 187). We will not go into details of motivation here, and refer the reader to, e.g., [Par96] for further discussion.

Thus, we have modular distances and relations, i.e., Hamming distances and relations, we have modular revision as described below, and we have modular logic, which has the (semantic) interpolation property. We want to point out here in particular this cross reference from modular revision to modular logic, i.e., logic with interpolation.

We recall:

Definition 5.2.4

Given a distance d, define for two sets X, Y

$X \mid Y := \{y \in Y : \exists x \in X(\neg\exists x' \in X, y' \in Y.d(x', y') < d(x, y))\}$.

We assume that $X \mid Y \neq \emptyset$ if $X, Y \neq \emptyset$. Note that this is related to the consistency axiom of AGM theory revision: revising by a consistent formula gives a consistent result. The assumption may be wrong due to infinite descending chains of distances.

Definition 5.2.5

Given $\mid$, we can define an AGM revision operator $*$ as follows:

$T * \phi := Th(M(T) \mid M(\phi))$

where T is a theory, and $Th(X)$ is the set of formulas which hold in all $x \in X$.

It was shown in [LMS01] that a revision operator thus defined satisfies the AGM revision postulates.

Definition 5.2.6

Let d be an abstract distance on some product space $X \times Y$, and its components. (We require of distances only that they are comparable, that $d(x,y) = 0$ iff $x = y$, and that $d(x,y) \geq 0$.)

d is called a generalized Hamming distance (GHD) iff it satisfies the following two properties:

$(GHD1)$ $d(\sigma,\tau) \leq d(\alpha,\beta)$ and $d(\sigma',\tau') \leq d(\alpha',\beta')$ and $(d(\sigma,\tau) < d(\alpha,\beta)$ or $d(\sigma',\tau') < d(\alpha',\beta'))$ $\Rightarrow d(\sigma\sigma',\tau\tau') < d(\alpha\alpha',\beta\beta')$

$(GHD2)$ $d(\sigma\sigma',\tau\tau') < d(\alpha\alpha',\beta\beta') \Rightarrow d(\sigma,\tau) < d(\alpha,\beta)$ or $d(\sigma',\tau') < d(\alpha',\beta')$

(Compare this definition to Definition 5.2.2 (page 181).)

We have a result analogous to the relation case:

Fact 5.2.5

Let $\mid$ be defined by a generalized Hamming distance, then $\mid$ satisfies

(1)

$(\mid *)$ $(\Sigma_1 \times \Sigma'_1) \mid (\Sigma_2 \times \Sigma'_2) = (\Sigma_1 \mid \Sigma_2) \times (\Sigma'_1 \mid \Sigma'_2)$.

(2) $(\Sigma'_1 \mid \Sigma'_2) \times (\Sigma''_1 \mid \Sigma''_2) \subseteq \Sigma_2$ and $(\Sigma'_2 \mid \Sigma'_1) \times (\Sigma''_2 \mid \Sigma''_1) \subseteq \Sigma_1 \Rightarrow (\Sigma_1) \mid (\Sigma_2) = (\Sigma'_1 \mid \Sigma'_2) \times (\Sigma''_1 \mid \Sigma''_2)$, if the distance is symmetric

(where Σ_i here is not necessarily $\Sigma'_i \times \Sigma''_i$, etc.).

Proof

(1) and (2).

"$\subseteq$":

Suppose $d(\sigma\sigma',\tau\tau')$ is minimal. If there is $\alpha \in \Sigma_1$, $\beta \in \Sigma_2$ such that $d(\alpha,\beta) < d(\sigma,\tau)$, then $d(\alpha\sigma',\beta\tau') < d(\sigma\sigma',\tau\tau')$ by $(GHD1)$, so $d(\sigma,\tau)$ and $d(\sigma',\tau')$ have to be minimal.

"$\supseteq$":

For the converse, suppose $d(\sigma,\tau)$ and $d(\sigma',\tau')$ are minimal, but $d(\sigma\sigma',\tau\tau')$ is not, so $d(\alpha\alpha',\beta\beta') < d(\sigma\sigma',\tau\tau')$ for some $\alpha\alpha',\beta\beta'$, then $d(\alpha,\beta) < d(\sigma,\tau)$ or $d(\alpha',\beta') < d(\sigma',\tau')$ by $(GHD2)$, contradiction.

□

These properties translate to logic as follows:

Corollary 5.2.6

If ϕ and ψ are defined on a separate language from that of ϕ' and ψ', and the distance satisfies $(GHD1)$ and $(GHD2)$, then for revision holds:

$(\phi \wedge \phi') * (\psi \wedge \psi') = (\phi * \psi) \wedge (\phi' * \psi')$.

Corollary 5.2.7

By Corollary 5.2.6 (page 186), Hamming distances generate decomposable revision operators a la Parikh, see [Par96], also in the generalized form of variable K and ϕ.

We conclude with a small result on partial (semantical) revision:

Fact 5.2.8

Let $\mid$ be defined by a Hamming distance, then:

$\Pi X \mid \Sigma \subseteq \Gamma \Rightarrow \Pi X' \mid (\Sigma \restriction X') \subseteq \Gamma \restriction X'$.

(Recall that Π' is the restriction of Π to X'.)

Proof

Let $t \in \Sigma \restriction X' - \Gamma \restriction X'$, we show $t \notin \Pi X' \mid (\Sigma \restriction X')$. Let $\tau \in \Sigma$ be such that $\tau' = t$, then $\tau \notin \Gamma$ (otherwise $t \in \Gamma \restriction X'$), so $\tau \notin \Pi X \mid \Sigma$, so there is $\alpha = \alpha'\alpha'' \in \Pi X$, $\beta = \beta'\beta'' \in \Sigma$, with $d(\alpha, \beta)$ minimal, so $d(\alpha, \beta) < d(\sigma, \tau)$ for all $\sigma \in \Pi X$. If $d(\sigma', \tau')$ were minimal for some σ, then we would consider $\sigma'\alpha''$, $\tau'\beta''$, then $d(\alpha'\alpha'', \beta'\beta'') < d(\sigma'\alpha'', \tau'\beta'')$ is impossible by (GHD2), so $\tau'\beta'' \in \Pi X \mid \Sigma$, so $\tau'\beta'' \in \Gamma$, and $t \in \Gamma \restriction X'$, contradiction.

$\square$

5.2.3.3 Discussion of representation

It would be nice to have a representation result like the one for Hamming relations, see Proposition 5.2.4 (page 182), (5). But this is impossible, for the following reason:

In constructing the representing distance from revision results, we made arbitrary choices (see the proofs in [LMS01] or [Sch04]). I.e., we choose sometimes arbitrarily $d(x, y) \leq d(x', y')$, when we do not have enough information to decide. (This is an example of the fact that the problem of "losing ignorance" should not be underestimated, see e.g. [GS08f].) As we do not follow the same procedure for all cases, there is no guarantee that the different representations will fit together.

Of course, it might be possible to come to a uniform choice, and one could then attempt a representation result. This is left as an open problem.

5.2.4 Summary of properties

We summarize in this section properties related to multiplicative laws.

They are collected in Table 5.3 (page 191).

$pr(b) = b$ means: the projection of a big set on one of its coordinates is big again.

Note that $A \times B \subseteq X \times Y$ big $\Rightarrow$ $A \subseteq X$ big etc. is intuitively better justified than the other direction, as the proportion might increase in the latter, decrease in the former. Cf. the table "Rules on size", Section 5.2.1 (page 171), "increasing proportions".

Table 5.1: Rules on size - Part I

Rules on size - Part I				
	"Ideal"	"Filter"	$\mathcal{M}^+$	∇
Optimal proportion				
(Opt)	$\emptyset \in \mathcal{I}(X)$	$X \in \mathcal{F}(X)$		$\forall x\alpha \rightarrow \nabla x\alpha$
Monotony (Improving proportions). (iM): internal monotony, $(eM\mathcal{I})$: external monotony for ideals, $(eM\mathcal{F})$: external monotony for filters				
(iM)	$A \subseteq B \in \mathcal{I}(X)$ $\Rightarrow$ $A \in \mathcal{I}(X)$	$A \in \mathcal{F}(X)$, $A \subseteq B \subseteq X$ $\Rightarrow B \in \mathcal{F}(X)$		$\nabla x\alpha \wedge \forall x(\alpha \rightarrow \alpha')$ $\rightarrow \nabla x\alpha'$
$(eM\mathcal{I})$	$X \subseteq Y \Rightarrow$ $\mathcal{I}(X) \subseteq \mathcal{I}(Y)$			$\nabla x(\alpha : \beta) \wedge$ $\forall x(\alpha' \rightarrow \beta) \rightarrow$ $\nabla x(\alpha \vee \alpha' : \beta)$
$(eM\mathcal{F})$		$X \subseteq Y \Rightarrow$ $\mathcal{F}(Y) \cap \mathcal{P}(X) \subseteq$ $\mathcal{F}(X)$		$\nabla x(\alpha : \beta) \wedge$ $\forall x(\beta \wedge \alpha \rightarrow \alpha') \rightarrow$ $\nabla x(\alpha \wedge \alpha' : \beta)$
Keeping proportions				
$(\approx)$	$(\mathcal{I} \cup disj)$ $A \in \mathcal{I}(X)$, $B \in \mathcal{I}(Y)$, $X \cap Y = \emptyset \Rightarrow$ $A \cup B \in \mathcal{I}(X \cup Y)$	$(\mathcal{F} \cup disj)$ $A \in \mathcal{F}(X)$, $B \in \mathcal{F}(Y)$, $X \cap Y = \emptyset \Rightarrow$ $A \cup B \in \mathcal{F}(X \cup Y)$	$(\mathcal{M}^+ \cup disj)$ $A \in \mathcal{M}^+(X)$, $B \in \mathcal{M}^+(Y)$, $X \cap Y = \emptyset \Rightarrow$ $A \cup B \in \mathcal{M}^+(X \cup Y)$	$\nabla x(\alpha : \beta) \wedge$ $\nabla x(\alpha' : \beta) \wedge$ $\neg \exists x(\alpha \wedge \alpha') \rightarrow$ $\nabla x(\alpha \vee \alpha' : \beta)$
Robustness of proportions: $n * small \neq All$				
$(1 * s)$	$(\mathcal{I}_1)$ $X \not\in \mathcal{I}(X)$	$(\mathcal{F}_1)$ $\emptyset \not\in \mathcal{F}(X)$		(∇_1) $\nabla x\alpha \rightarrow \exists x\alpha$
$(2 * s)$	$(\mathcal{I}_2)$ $A, B \in \mathcal{I}(X) \Rightarrow$ $A \cup B \neq X$	$(\mathcal{F}_2)$ $A, B \in \mathcal{F}(X) \Rightarrow$ $A \cap B \neq \emptyset$		(∇_2) $\nabla x\alpha \wedge \nabla x\beta$ $\rightarrow \exists x(\alpha \wedge \beta)$
$(n * s)$ $(n \geq 3)$	$(\mathcal{I}_n)$ $A_1, ., A_n \in \mathcal{I}(X)$ $\Rightarrow$ $A_1 \cup . \cup A_n \neq X$	$(\mathcal{F}_n)$ $A_1, ., A_n \in \mathcal{I}(X)$ $\Rightarrow$ $A_1 \cap . \cap A_n \neq \emptyset$	$(\mathcal{M}_n^+)$ $X_1 \in \mathcal{F}(X_2), .,$ $X_{n-1} \in \mathcal{F}(X_n) \Rightarrow$ $X_1 \in \mathcal{M}^+(X_n)$	(∇_n) $\nabla x\alpha_1 \wedge . \wedge \nabla x\alpha_n$ $\rightarrow$ $\exists x(\alpha_1 \wedge . \wedge \alpha_n)$
$(< \omega * s)$	$(\mathcal{I}_\omega)$ $A, B \in \mathcal{I}(X) \Rightarrow$ $A \cup B \in \mathcal{I}(X)$	$(\mathcal{F}_\omega)$ $A, B \in \mathcal{F}(X) \Rightarrow$ $A \cap B \in \mathcal{F}(X)$	$(\mathcal{M}_\omega^+)$ (1) $A \in \mathcal{F}(X),\ X \in \mathcal{M}^+(Y)$ $\Rightarrow A \in \mathcal{M}^+(Y)$ (2) $A \in \mathcal{M}^+(X),\ X \in \mathcal{F}(Y)$ $\Rightarrow A \in \mathcal{M}^+(Y)$ (3) $A \in \mathcal{F}(X),\ X \in \mathcal{F}(Y)$ $\Rightarrow A \in \mathcal{F}(Y)$ (4) $A, B \in \mathcal{I}(X) \Rightarrow$ $A - B \in \mathcal{I}(X - \mathrm{B})$	(∇_ω) $\nabla x\alpha \wedge \nabla x\beta \rightarrow$ $\nabla x(\alpha \wedge \beta)$
Robustness of $\mathcal{M}^+$				
$(\mathcal{M}^{++})$			$(\mathcal{M}^{++})$ (1) $A \in \mathcal{I}(X),\ B \not\in \mathcal{F}(X)$ $\Rightarrow A - B \in \mathcal{I}(X - B)$ (2) $A \in \mathcal{F}(X), B \not\in \mathcal{F}(X)$ $\Rightarrow A - B \in \mathcal{F}(X - B)$ (3) $A \in \mathcal{M}^+(X)$, $X \in \mathcal{M}^+(Y)$ $\Rightarrow A \in \mathcal{M}^+(Y)$	

Table 5.2: Rules on size - Part II

Rules on size - Part II				
	various rules	AND	OR	Caut./Rat.Mon.
Optimal proportion				
(Opt)	(SC) $\alpha \vdash \beta \Rightarrow \alpha \mid\sim \beta$			
Monotony (Improving proportions)				
(iM)	(RW) $\alpha \mid\sim \beta, \beta \vdash \beta' \Rightarrow$ $\alpha \mid\sim \beta'$			
$(eM\mathcal{I})$	(PR') $\alpha \mid\sim \beta, \alpha \vdash \alpha'$, $\alpha' \wedge \neg\alpha \vdash \beta \Rightarrow$ $\alpha' \mid\sim \beta$ (μPR) $X \subseteq Y \Rightarrow$ $\mu(Y) \cap X \subseteq \mu(X)$		(wOR) $\alpha \mid\sim \beta,\ \alpha' \vdash \beta \Rightarrow$ $\alpha \vee \alpha' \mid\sim \beta$ (μwOR) $\mu(X \cup Y) \subseteq \mu(X) \cup Y$	
$(eM\mathcal{F})$				(wCM) $\alpha \mid\sim \beta, \alpha' \vdash \alpha$, $\alpha \wedge \beta \vdash \alpha' \Rightarrow$ $\alpha' \mid\sim \beta$
Keeping proportions				
$(\approx)$	(NR) $\alpha \mid\sim \beta \Rightarrow$ $\alpha \wedge \gamma \mid\sim \beta$ or $\alpha \wedge \neg\gamma \mid\sim \beta$		$(disjOR)$ $\alpha \mid\sim \beta,\ \alpha' \mid\sim \beta'$ $\alpha \vdash \neg\alpha', \Rightarrow$ $\alpha \vee \alpha' \mid\sim \beta \vee \beta'$ $(\mu disjOR)$ $X \cap Y = \emptyset \Rightarrow$ $\mu(X \cup Y) \subseteq \mu(X) \cup \mu(Y)$	
Robustness of proportions: $n * small \neq All$				
$(1 * s)$	(CP) $\alpha \mid\sim \bot \Rightarrow \alpha \vdash \bot$	(AND_1) $\alpha \mid\sim \beta \Rightarrow \alpha \not\vdash \neg\beta$		
$(2 * s)$		(AND_2) $\alpha \mid\sim \beta,\ \alpha \mid\sim \beta' \Rightarrow$ $\alpha \not\vdash \neg\beta \vee \neg\beta'$	(OR_2) $\alpha \mid\sim \beta \Rightarrow \alpha \not\mid\sim \neg\beta$	(CM_2) $\alpha \mid\sim \beta \Rightarrow \alpha \not\mid\sim \neg\beta$
$(n * s)$ $(n \geq 3)$		(AND_n) $\alpha \mid\sim \beta_1, ., \alpha \mid\sim \beta_n$ $\Rightarrow$ $\alpha \not\vdash \neg\beta_1 \vee . \vee \neg\beta_n$	(OR_n) $\alpha_1 \mid\sim \beta, ., \alpha_{n-1} \mid\sim \beta$ $\Rightarrow$ $\alpha_1 \vee . \vee \alpha_{n-1} \not\mid\sim \neg\beta$	(CM_n) $\alpha \mid\sim \beta_1, ., \alpha \mid\sim \beta_{n-1}$ $\Rightarrow$ $\alpha \wedge \beta_1 \wedge . \wedge \beta_{n-2} \not\mid\sim$ $\neg\beta_{n-1}$
$(< \omega * s)$		(AND_ω) $\alpha \mid\sim \beta,\ \alpha \mid\sim \beta' \Rightarrow$ $\alpha \mid\sim \beta \wedge \beta'$	(OR_ω) $\alpha \mid\sim \beta,\ \alpha' \mid\sim \beta \Rightarrow$ $\alpha \vee \alpha' \mid\sim \beta$ (μOR) $\mu(X \cup Y) \subseteq \mu(X) \cup \mu(Y)$	(CM_ω) $\alpha \mid\sim \beta,\ \alpha \mid\sim \beta' \Rightarrow$ $\alpha \wedge \beta \mid\sim \beta'$ (μCM) $\mu(X) \subseteq Y \subseteq X \Rightarrow$ $\mu(Y) \subseteq \mu(X)$
Robustness of $\mathcal{M}^+$				
$(\mathcal{M}^{++})$				$(RatM)$ $\alpha \mid\sim \beta, \alpha \not\mid\sim \neg\beta' \Rightarrow$ $\alpha \wedge \beta' \mid\sim \beta$ $(\mu RatM)$ $X \subseteq Y$, $X \cap \mu(Y) \neq \emptyset \Rightarrow$ $\mu(X) \subseteq \mu(Y) \cap X$

.

Multiplication laws								
Multiplication law	Scenario 1 (see Diagram 5.2.1 (page 174))			Scenario 2 ($*$ symmetrical, only 1 side shown) (see Diagram 5.2.2 (page 176))				
	Corresponding algebraic addition property	Logical property	Relation property	Algebraic property ($\Gamma_i \subseteq \Sigma_i$)	Logical property (α, β in L_1, α', β' in L_2 $L = L_1 \cup L_2$ (disjoint))	Interpolation		
						Multiplic. law	Relation property	Inter-polation
Non-monotonic logic								
$x * 1 \Rightarrow x$	trivial			$\Gamma_1 \in \mathcal{F}(\Sigma_1) \Rightarrow$ $\Gamma_1 \times \Sigma_2 \in \mathcal{F}(\Sigma_1 \times \Sigma_2)$	$\alpha \mid\!\sim_{L_i} \beta \Rightarrow \alpha \mid\!\sim_L \beta$			
$1 * x \Rightarrow x$	trivial							
$x * s \Rightarrow s$	(iM) $A \subseteq B \in \mathcal{I}(X) \Rightarrow A \in \mathcal{I}(X)$	$\alpha \mid\!\sim \neg\beta \Rightarrow$ $\alpha \mid\!\sim \neg\beta \vee \gamma$	-	$\Gamma_1 \in I(\Sigma_1) \Rightarrow$ $\Gamma_1 \times \Gamma_2 \in I(\Sigma_1 \times \Sigma_2)$	$\alpha \mid\!\sim_{L_1} \beta,\ \beta' \vdash_{L_2} \alpha' \Rightarrow$ $\alpha \wedge \alpha' \mid\!\sim (\beta \wedge \alpha') \vee (\alpha \wedge \beta')$			
$s * x \Rightarrow s$	$(eM\mathcal{I})$ $X \subseteq Y \Rightarrow \mathcal{I}(X) \subseteq \mathcal{I}(Y)$, $X \subseteq Y \Rightarrow$ $\mathcal{F}(Y) \cap \mathcal{P}(X) \subseteq \mathcal{F}(X)$	$\alpha \wedge \beta \mid\!\sim \neg\gamma \Rightarrow$ $\alpha \mid\!\sim \neg\beta \vee \neg\gamma$	-					
$b * b \Rightarrow b$ $(\mu * 1)$	$(< \omega * s)$, $(\mathcal{M}^+_\omega)$ (3) $A \in \mathcal{F}(X), X \in \mathcal{F}(Y) \Rightarrow$ $A \in \mathcal{F}(Y)$	$\alpha \mid\!\sim \beta,\ \alpha \wedge \beta \mid\!\sim \gamma$ $\Rightarrow \alpha \mid\!\sim \gamma$	- (Filter)	$\Gamma_1 \in \mathcal{F}(\Sigma_1), \Gamma_2 \in \mathcal{F}(\Sigma_2) \Rightarrow$ $\Gamma_1 \times \Gamma_2 \in \mathcal{F}(\Sigma_1 \times \Sigma_2)$	$\alpha \mid\!\sim_{L_1} \beta,\ \alpha' \mid\!\sim_{L_2} \beta' \Rightarrow$ $\alpha \wedge \alpha' \mid\!\sim_L \beta \wedge \beta'$	$b * b \Leftrightarrow b$ $(\mu * 1)$	(GH)	$\mid\!\sim \circ \mid\!\sim$
$b * m \Rightarrow m$	$(< \omega * s)$, $(\mathcal{M}^+_\omega)$ (2) $A \in \mathcal{M}^+(X), X \in \mathcal{F}(Y) \Rightarrow$ $A \in \mathcal{M}^+(Y)$	$\alpha \mid\!\sim \beta,\ \alpha \wedge \beta \not\mid\!\sim \neg\gamma$ $\Rightarrow \alpha \not\mid\!\sim \neg\beta \vee \neg\gamma$	- (Filter)	$\Gamma_1 \in \mathcal{F}(\Sigma_1), \Gamma_2 \in \mathcal{M}^+(\Sigma_2) \Rightarrow$ $\Gamma_1 \times \Gamma_2 \in \mathcal{M}^+(\Sigma_1 \times \Sigma_2)$	$\alpha \not\mid\!\sim_{L_1} \neg\beta,\ \alpha' \mid\!\sim_{L_2} \beta' \Rightarrow$ $\alpha \wedge \alpha' \not\mid\!\sim_L \neg(\beta \wedge \beta')$			
$m * b \Rightarrow m$	$(< \omega * s)$, $(\mathcal{M}^+_\omega)$ (1) $A \in \mathcal{F}(X), X \in \mathcal{M}^+(Y) \Rightarrow$ $A \in \mathcal{M}^+(Y)$	$\alpha \not\mid\!\sim \neg\beta,\ \alpha \wedge \beta \mid\!\sim \gamma$ $\Rightarrow \alpha \not\mid\!\sim \neg\beta \vee \neg\gamma$	- (Filter)					
$m * m \Rightarrow m$	$(\mathcal{M}^{++})$ $A \in \mathcal{M}^+(X), X \in \mathcal{M}^+(Y)$ $\Rightarrow A \in \mathcal{M}^+(Y)$	Rational Monotony	ranked	$\Gamma_1 \in \mathcal{M}^+(\Sigma_1), \Gamma_2 \in \mathcal{M}^+(\Sigma_2)$ $\Rightarrow$ $\Gamma_1 \times \Gamma_2 \in \mathcal{M}^+(\Sigma_1 \times \Sigma_2)$	$\alpha \not\mid\!\sim_{L_1} \neg\beta,\ \alpha' \not\mid\!\sim_{L_2} \neg\beta' \Rightarrow$ $\alpha \wedge \alpha' \not\mid\!\sim_L \neg(\beta \wedge \beta')$			
$b * b \Leftrightarrow b$, $pr(b) = b$ $(\mu * 2)$					$\alpha \mid\!\sim \beta \Rightarrow \alpha \upharpoonright L_1 \mid\!\sim \beta \upharpoonright L_1$ and $\alpha \mid\!\sim_{L_1} \beta,\ \alpha' \mid\!\sim_{L_2} \beta' \Rightarrow$ $\alpha \wedge \alpha' \mid\!\sim_L \beta \wedge \beta'$	$(\mu * 1)$ + $(\mu * 2)$	(GH3)	$\vdash \circ \mid\!\sim$
J' small					$\alpha \wedge \alpha' \mid\!\sim \beta \wedge \beta' \Leftrightarrow$ $\alpha \mid\!\sim \beta,\ \alpha' \vdash \beta'$		$forget(J')$	-
Theory revision								
				$(\mid *)$: $(\Sigma_1 \times \Sigma_1') \mid (\Sigma_2 \times \Sigma_2') =$ $(\Sigma_1 \mid \Sigma_2) \times (\Sigma_1' \mid \Sigma_2')$	$(\phi \wedge \phi') * (\psi \wedge \psi') =$ $(\phi * \psi) \wedge (\phi' * \psi')$		(GHD)	$(\phi \wedge \phi') * (\psi \wedge \psi') \vdash \rho$ $\Rightarrow \phi' * \psi' \vdash \rho$ ϕ, ψ in J, ϕ', ψ', ρ in $L - J$

Table 5.3: Multiplication laws

5.2.5 Language change in classical and non-monotonic logic

Fact 5.2.9

We can obtain factorization by language change, provided cardinalities permit this.

Proof

Consider k variables, suppose we have $p = m * n$ positive instances, and that we can divide k into k' and k'' such that $2^{k'} \geq m$, $2^{k''} \geq n$, then we can factorize:

Choose m sequences of 0/1 of length k', n sequences of length k''. They will code the positive instances: there are $p = m * n$ pairs of the chosen sequences. Take any bijection between these pairs and the positive instances, and continue the bijection arbitrarily between other pairs and negative instances.

We can do the same also for 2 sets, corresponding to K, ϕ to have a common factorization, they both have to admit common factors like m, n above. We then choose first the pairs for e.g. K, then for ϕ, then the rest.

□

Example 5.2.5

Consider $p = 3$, and let

abc, $a\neg bc$, $a\neg b\neg c$, $\neg abc$, $\neg a\neg b\neg c$, $\neg ab\neg c$ be the $6 = 2 * 3$ positive cases,

$ab\neg c$, $\neg a\neg bc$ the negative ones. (It is coincidence that we can factorize positive and negative cases - probably iff one of the factors is the full product, here 2, it could also be 4 etc.)

We divide the cases by 3 new variables, grouping them together in positive and negative cases. a' is indifferent, we want this to be the independent factor, the negative ones will be put into $\neg b'\neg c'$. The procedure has to be made precise still. (n): negative

Let a' code the set abc, $a\neg bc$, $a\neg b\neg c$, $ab\neg c$ (n),

Let $\neg a'$ code $\neg a\neg bc$ (n), $\neg abc$, $\neg a\neg b\neg c$, $\neg ab\neg c$.

Let b' code abc, $a\neg bc$, $\neg a\neg b\neg c$, $\neg ab\neg c$

Let $\neg b'$ code $a\neg b\neg c$, $ab\neg c$ (n), $\neg a\neg bc$ (n), $\neg abc$

Let c' code abc, $a\neg b\neg c$, $\neg abc$, $\neg a\neg b\neg c$

Let $\neg c'$ code $a\neg bc$, $ab\neg c$ (n), $\neg a\neg bc$ (n), $\neg ab\neg c$

Then the 6 positive instances are

$\{a', \neg a'\} \times \{b'c',\ b'\neg c',\ \neg b'c'\}$, the negative ones

$\{a', \neg a'\} \times \{\neg b'\neg c'\}$

As we have 3 new variables, we code again all possible cases, so expressivity is the same.

□

The same holds for non-monotonic logic.

We give an example:

Example 5.2.6

Suppose we have the rule that "positive is better than negative" (all other things equal). Then, for two variables, a and b, we have the comparisons $ab \prec a\neg b \prec \neg a\neg b$, and $ab \prec \neg ab \prec \neg a\neg b$.

Suppose now we are given the situation $\neg cd \prec c\neg d \prec cd$ and $\neg cd \prec \neg c\neg d \prec cd$, which has the same order structure, but with negations not fitting. We put $c\neg d$ and $\neg cd$ into a new variable a', cd and $\neg c\neg d$ into $\neg a'$, $\neg cd$ and $\neg c\neg d$ into b', and $c\neg d$ and cd into $\neg b'$. Then $a'b'$ corresponds to $\neg cd$, $a'\neg b'$ to $c\neg d$, $\neg a'b'$ to $\neg c\neg d$, $\neg a'\neg b'$ to cd - and we have the desired structure. Thus, if the geometric structure is possible, then we can change language and obtain the desired pattern.

But we cannot obtain by language change a pattern of the type $ab \prec a\neg b$ without any other comparison, if it is supposed to be based on a component-wise comparison.

We summarize:

We can cut the model set as we like:

Choose half to go into p_0, half into $\neg p_0$, again half of p_0 into $p_0 \wedge p_1$, half into $p_0 \wedge \neg p_1$, etc.

5.3 Semantic interpolation for non-monotonic logic

5.3.1 Discussion

We discuss here the full non-monotonic case, i.e., downward and upward. We consider here a non-monotonic logic $\mid\sim$. We look at the interpolation problem in 3 ways.

Given $\phi \mid\sim \psi$, there is an interpolant α such that

(1) $\phi \mid\sim \alpha \vdash \psi$, see Section 5.3.2 (page 194),

(2) $\phi \vdash \alpha \mid\sim \psi$, see Section 5.3.3 (page 196),

(3) $\phi \mid\sim \alpha \mid\sim \psi$, see Section 5.3.4 (page 198).

The first variant will be fully characterized below.

The second and third variant have no full characterization at the time of writing (to the authors' knowledge), but are connected to very interesting properties about multiplication of size and componentwise independent relations.

We begin with the following negative result:

Example 5.3.1

Full non-monotonic logics, i.e. down and up, has not necessarily interpolation.

Consider the model order $pq \prec p\neg q \prec \neg p\neg q \prec \neg pq$. Then $\neg p \mid\sim \neg q$, there are no common variables, and $true \mid\sim q$ (and, of course, $\neg p \not\mid\sim false$). (Full consequence of $\neg p$ is $\neg p\neg q$, so this has trivial interpolation.)

5.3.2 Interpolation of the form $\phi \mid\sim \alpha \vdash \psi$

Fact 5.3.1

Let $var(\phi)$ be the set of relevant variables of ϕ.

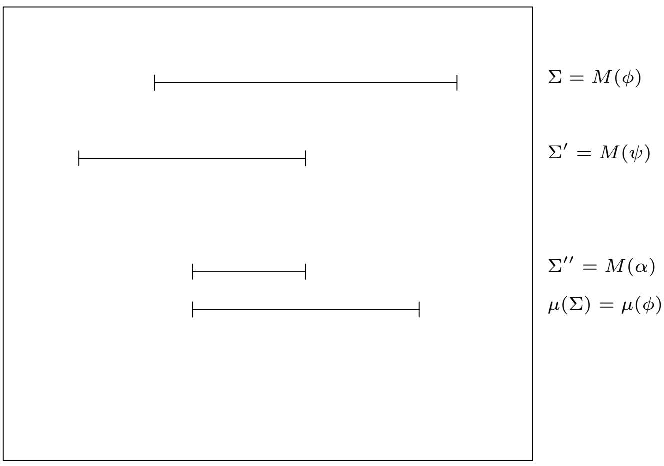
Let $\Sigma \subseteq \Pi X$, $var(\alpha) \cap var(\beta) = \emptyset$, $var(\beta) \cap R(\Sigma) = \emptyset$, β not a tautology, then $\Sigma \subseteq M(\alpha \vee \beta) \Rightarrow \Sigma \subseteq M(\alpha)$.

Proof

Suppose not, so there is $\sigma \in \Sigma$ such that $\sigma \models \alpha \vee \beta$, $\sigma \not\models \alpha$. As β is not a tautology, there is an assignment to $var(\beta)$ which makes β wrong. Consider τ such that $\sigma = \tau$ except on $var(\beta)$, where τ makes β wrong, using this assignment. By $var(\alpha) \cap var(\beta) = \emptyset$, $\tau \models \neg\alpha$. By $var(\beta) \cap R(\Sigma) = \emptyset$, $\tau \in \Sigma$. So $\tau \not\models \alpha \vee \beta$ for some $\tau \in \Sigma$, contradiction.

□

Diagram 5.3.1



Non-monotonic interpolation, $\phi \mid\sim \alpha \vdash \psi$

Proposition 5.3.2

We use here normal forms (conjunctions of disjunctions).

Consider a finite language. Let a semantic choice function μ be given, as discussed in Section 1.5.1 (page 19) , defined for sets of sequences (read: models).

$\mid\sim$ has interpolation iff for all Σ $I(\Sigma) \subseteq I(\mu(\Sigma))$ holds.

In the infinite case, we need as additional prerequisite that $\mu(\Sigma)$ is definable if Σ is.

Proof

Work with reformulations of Σ etc. which use only essential (= relevant) variables.

"$\Rightarrow$":

Suppose the condition is wrong. Then $X := I(\Sigma) - I(\mu(\Sigma)) = I(\Sigma) \cap R(\mu(\Sigma)) \neq \emptyset$. Thus there is some $\sigma' \in \mu(\Sigma) \upharpoonright R(\Sigma)$ which cannot be continued by some choice ρ in $X \cup (I(\Sigma) \cap I(\mu(\Sigma)))$ in $\mu(\Sigma)$, i.e. $\sigma'\rho \notin \mu(\Sigma)$.

We first do the finite case: We identify models with their describing formulas. Consider the formula $\phi := \sigma' \rightarrow \neg\rho = \neg\sigma' \vee \neg\rho$. We have $Th(\Sigma) \mid\sim \phi$, as $\mu(\Sigma) \subseteq M(\phi)$. Suppose Σ'' is a semantical interpolant for Σ and ϕ. So $\mu(\Sigma) \subseteq \Sigma'' \subseteq M(\phi)$, and Σ'' does not contain any variables in ρ as essential variables. By Fact 5.3.1 (page 194), $\mu(\Sigma) \subseteq \Sigma'' \subseteq M(\neg\sigma')$, but $\sigma' \in \mu(\Sigma) \upharpoonright R(\Sigma)$, contradiction.

We turn to the infinite case. Consider again $\sigma'\rho$. As $\sigma'\rho \notin \mu(\Sigma)$, and $\mu(\Sigma)$ is definable, there is some formula ϕ which holds in $\mu(\Sigma)$, but fails in $\sigma'\rho$. Thus, $Th(\Sigma) \mid\sim \phi$. Write ϕ as a disjunction of conjunctions. Let Σ'' be an interpolant for Σ and $M(\phi)$. Thus $\mu(\Sigma) \subseteq \Sigma'' \subseteq M(\phi)$, and $\sigma'\rho \notin M(\phi)$, so $\mu(\Sigma) \subseteq \Sigma'' \subseteq M(\phi) \subseteq M(\neg\sigma' \vee \neg\rho)$, so $\Sigma'' \subseteq M(\neg\sigma')$ by Fact 5.3.1 (page 194). So $\mu(\Sigma) \models \neg\sigma'$, contradiction, as $\sigma' \in \mu(\Sigma) \upharpoonright R(\Sigma)$. (More precisely, we have to argue here with not necessarily definable model sets.)

"$\Leftarrow$":

Let $I(\Sigma) \subseteq I(\mu(\Sigma))$. Let $\Sigma \mid\sim \Sigma'$, i.e. $\mu(\Sigma) \subseteq \Sigma'$. Write $\mu(\Sigma)$ as a (possibly infinite) conjunction of disjunctions, using only relevant variables. Form Σ'' from $\mu(\Sigma)$ by omitting all variables in this description which are not in $R(\Sigma')$. Note that all remaining variables are in $R(\mu(\Sigma)) \subseteq R(\Sigma)$, so Σ'' is a candidate for interpolation. See Diagram 5.3.1 (page 194).

(1) $\mu(\Sigma) \subseteq \Sigma''$: Trivial.

(2) $\Sigma'' \subseteq \Sigma'$: Let $\sigma \in \Sigma''$. Then there is $\tau \in \mu(\Sigma) \subseteq \Sigma'$ such that $\sigma \upharpoonright R(\Sigma') = \tau \upharpoonright R(\Sigma')$, so $\sigma \in \Sigma'$.

A shorter argument is as follows: $\mu(\Sigma) \models M(\Sigma')$ has a semantic interpolant by Section 4.2 (page 130), which is by prerequisite also an interpolant for Σ and Σ'.

It remains to show in the infinite case that Σ'' is definable. This can be shown as in Proposition 4.4.1 (page 147).

□

5.3.3 Interpolation of the form $\phi \vdash \alpha \mid\sim \psi$

This situation is much more interesting than the last one, discussed in Section 5.3.2 (page 194). In this section, and the next one, Section 5.3.4 (page 198), we connect abstract multiplication laws for size to interpolation. To our knowledge, such multiplication laws are considered here for the first time, and so also their connection to interpolation problems.

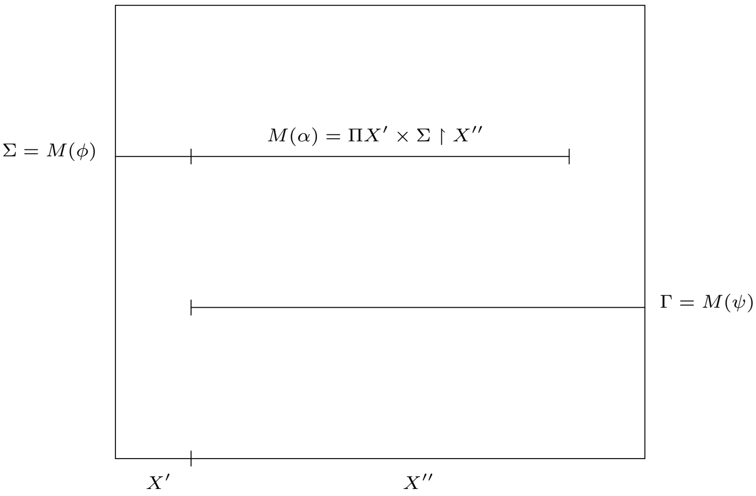
We introduced two sets of three conditions about abstract size (see Definition 5.2.1 (page 178)), and then showed in Proposition 5.2.1 (page 179) that both sets are equivalent.

We show now that the first two, or the last condition entail interpolation, see Proposition 5.3.3 (page 197).

Recall that, in preferential structures, size is generated by a relation. $A \subseteq B$ is a big subset iff A contains all minimal elements of B (with respect to this relation). Hamming relations, see Definition 5.2.2 (page 181), generate a notion of size which satisfies our multiplicative conditions (if they are smooth).

Thus, if a preferential logic is defined by a smooth Hamming relation, it has semantic interpolation (in our sense here). This is summarized in Corollary 5.3.4 (page 198).

Diagram 5.3.2



Non-monotonic interpolation, $\phi \vdash \alpha \mid\!\sim \psi$

Proposition 5.3.3

We assume definability as shown in Proposition 4.4.1 (page 147).

Interpolation of the form $\phi \vdash \alpha \mid\!\sim \psi$ exists, if

(1) both $(S*1)$ and $(S*2)$,

or

(2) $(S*3)$ hold,

when $\beta \mid\!\sim \gamma$ is defined by:

$\beta \mid\!\sim \gamma :\Leftrightarrow \mu(\beta) = \mu(M(\beta)) \subseteq M(\gamma)$, and

$\mu(X)$ is the generator of the principal filter over X.

(We saw in Example 5.2.1 (page 181) that $(S*2)$, and thus also $(S*3)$, will often be too strong.)

Proof

Let $\Sigma := M(\phi)$, $\Gamma := M(\psi)$, X' the set of variables only in ϕ, so $\Gamma = \Pi X' \times \Gamma \upharpoonright X''$, where $\Pi X' = \Pi X'$. Set $\alpha := Th(\Pi X' \times \Sigma'')$, where $\Sigma'' = \Sigma \upharpoonright X''$. Note that variables only in ψ are automatically taken care of, as Σ'' can be written as a product without mentioning them. See Diagram 5.3.2 (page 196).

By prerequisite, $\mu(\Sigma) \subseteq \Gamma$, we have to show $\mu(\Pi X' \times \Sigma'') \subseteq \Gamma$.

(1)

$\mu(\Pi X' \times \Sigma'') = \mu(\Pi X') \times \mu(\Sigma'')$ by $(S * 1)$ and Proposition 5.2.1 (page 179) (2). By $\mu(\Sigma) \subseteq \Gamma$, $(S * 2)$, and Proposition 5.2.1 (page 179) (2), $\mu(\Sigma'') = \mu(\Sigma \restriction X'') \subseteq \Gamma \restriction X''$, so $\mu(\Pi X' \times \Sigma'') = \mu(\Pi X') \times \mu(\Sigma'') \subseteq \mu(\Pi X') \times \Gamma \restriction X'' \subseteq \Gamma$.

(2)

$\mu(\Pi X' \times \Sigma'') \restriction X'' \subseteq \mu(\Sigma) \restriction X'' \subseteq \Gamma \restriction X''$ by $(S * 3)$ and Proposition 5.2.1 (page 179) (2). So $\mu(\Pi X' \times \Sigma'') \subseteq \Pi X' \times (\mu(\Pi X' \times \Sigma'') \restriction X'') \subseteq \Pi X' \times (\Gamma \restriction X'') = \Gamma$.

□

The following Corollary puts our results together.

Corollary 5.3.4

Interpolation in the form $\phi \vdash \alpha \mid\sim \psi$ exists, when $\mid\sim$ is defined by a smooth Hamming relation, more precisely, a smooth relation satisfying (GH3).

Proof

We give two proofs:

(1)

By Proposition 5.2.4 (page 182) $(S * 1)$ and $(S * 2)$ hold. Thus, by Proposition 5.3.3 (page 197) (1), interpolation exists.

(2)

By Proposition 5.2.4 (page 182), $(S * 3)$ holds, so by Proposition 5.3.3 (page 197) (2), interpolation exists.

□

5.3.4 Interpolation of the form $\phi \mid\sim \alpha \mid\sim \psi$

The following result is perhaps the main result of the book. The conditions are natural, and not too strong, and the connection between those multiplicative properties and interpolation gives quite deep insights into the basics of non-monotonic logics.

Proposition 5.3.5

$(\mu * 1)$ entails semantical interpolation of the form $\phi \mid\sim \alpha \mid\sim \psi$ in 2-valued non-monotonic logic generated by minimal model sets. (As the model sets might not be definable, syntactic interpolation does not follow automatically.)

Proof

Let the product be defined on $J \cup J' \cup J''$ (i.e., $J \cup J' \cup J''$ is the set of propositional variables in the intended application). Let ϕ be defined on $J \cup J'$, ψ on $J' \cup J''$. See Diagram 5.3.3 (page 199).

We abuse notation and write $\phi \mid\!\sim \Sigma$ if $\mu(\phi) \subseteq \Sigma$. As usual, $\mu(\phi)$ abbreviates $\mu(M(\phi))$.

For clarity, even if it clutters up notation, we will be precise about where μ is formed. Thus, we write $\mu_{J\cup J'\cup J''}(X)$ when we take the minimal elements in the full product, $\mu_J(X)$ when we consider only the product on J, etc. X_J will be shorthand for $\Pi\{X_j : j \in J\}$.

Let $\phi \mid\!\sim \psi$, i.e., $\mu_{J\cup J'\cup J''}(\phi) \subseteq M(\psi)$. We show that $X_J \times (\mu_{J\cup J'\cup J''}(\phi) \restriction J') \times X_{J''}$ is a semantical interpolant, i.e., that $\mu_{J\cup J'\cup J''}(\phi) \subseteq X_J \times (\mu_{J\cup J'\cup J''}(\phi) \restriction J') \times X_{J''}$, and that $\mu_{J\cup J'\cup J''}(X_J \times (\mu_{J\cup J'\cup J''}(\phi) \restriction J') \times X_{J''}) \subseteq M(\psi)$.

The first property is trivial, we turn to the second.

(1) As $M(\phi) = M(\phi) \restriction (J \cup J') \times X_{J''}$, $\mu_{J\cup J'\cup J''}(\phi) = \mu_{J\cup J'}(M(\phi) \restriction (J \cup J')) \times \mu_{J''}(X_{J''})$ by $(\mu * 1)$.

(2) By $(\mu * 1)$ again, $\mu_{J\cup J'\cup J''}(X_J \times (\mu_{J\cup J'\cup J''}(\phi) \restriction J') \times X_{J''}) = \mu_J(X_J) \times \mu_{J'}(\mu_{J\cup J'\cup J''}(\phi) \restriction J') \times \mu_{J''}(X_{J''})$.

So it suffices to show $\mu_J(X_J) \times \mu_{J'}(\mu_{J\cup J'\cup J''}(\phi) \restriction J') \times \mu_{J''}(X_{J''}) \models \psi$.

Proof: Let $\sigma = \sigma_J \sigma_{J'} \sigma_{J''} \in \mu_J(X_J) \times \mu_{J'}(\mu_{J\cup J'\cup J''}(\phi) \restriction J') \times \mu_{J''}(X_{J''})$, so $\sigma_J \in \mu_J(X_J)$.

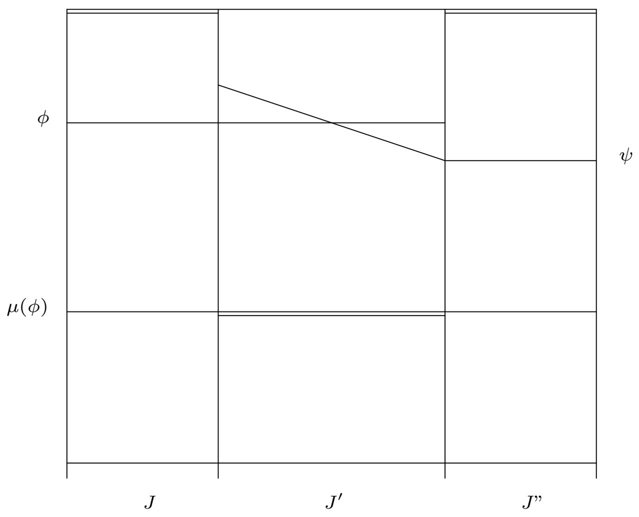
By $\mu_{J'}(\mu_{J\cup J'\cup J''}(\phi) \restriction J') \subseteq \mu_{J\cup J'\cup J''}(\phi) \restriction J'$, there is $\sigma' = \sigma'_J \sigma'_{J'} \sigma'_{J''} \in \mu_{J\cup J'\cup J''}(\phi)$ such that $\sigma'_{J'} = \sigma_{J'}$, i.e. $\sigma' = \sigma'_J \sigma_{J'} \sigma'_{J''}$. As $\sigma' \in \mu_{J\cup J'\cup J''}(\phi)$, $\sigma' \models \psi$.

By (1) and $\sigma_{J''} \in \mu_{J''}(X_{J''})$ also $\sigma'_J \sigma_{J'} \sigma_{J''} \in \mu_{J\cup J'\cup J''}(\phi)$, so also $\sigma'_J \sigma_{J'} \sigma_{J''} \models \psi$.

But ψ does not depend on J, so also $\sigma = \sigma_J \sigma_{J'} \sigma_{J''} \models \psi$.

□

Diagram 5.3.3



Non-monotonic interpolation, $\phi \mid\sim \alpha \mid\sim \psi$
Double lines: interpolant $\Pi J \times (\mu(\phi) \upharpoonright J') \times \Pi J$"
Alternative interpolants (in center part): $\phi \upharpoonright J'$ *or* $(\phi \wedge \psi) \upharpoonright J'$

5.3.4.1 Remarks for the converse: from interpolation to $(\mu * 1)$

Example 5.3.2

We show here in (1.1) and (1.2) that half of the condition $(\mu * 1)$ is not sufficient for interpolation, and in (2) that interpolation may hold, even if $(\mu * 1)$ fails. When looking closer, the latter is not surprising: μ of sub-products may be defined in a funny way, which has nothing to do with the way μ on the big product is defined.

Consider the language based on p, q, r. We define two orders:

(a) $\prec$ on sequences of length 3 by $\neg p \neg q \neg r \prec p \neg q \neg r$, and leave all other sequences of length 3 $\prec -$incomparabel.

(b) $<$ on sequences of the same length by $\sigma < \tau$ iff there is $\neg x$ is σ, x in τ, but for no y y in σ, $\neg y$ in τ. E.g., $\neg p < p$, $\neg pq < pq$, $\neg p \neg q < pq$, but $\neg pq \not< p \neg q$.

Work now with $\prec$. Let $\phi = \neg q \wedge \neg r$, $\psi = \neg p \wedge \neg q$, so $\mu(\phi) = \neg p \wedge \neg q \wedge \neg r$, and $\phi \mid\sim \psi$. Suppose there is α, $\phi \mid\sim \alpha \mid\sim \psi$, α written with q only, so α is equivalent to FALSE, TRUE, q, or $\neg q$. $\phi \not\mid\sim$ $FALSE$, $\phi \not\mid\sim$ q. $TRUE \not\mid\sim$ ψ, $\neg q \not\mid\sim$ ψ. Thus, there is no such α, and $\mid\sim$ has no interpolation.

We show in (1.1) and (1.2) that we can make both directions of $(\mu * 1)$ true separately, so they do not suffice to obtain interpolation.

(1.1) We make $\mu(X \times Y) \subseteq \mu(X) \times \mu(Y)$ true, but not the converse. Take the order $\prec$ on sequences of length 3 as described above. Do not order any sequences of length 2 or 1, i.e. μ is there always identity. Thus, $\mu(X \times Y) \subseteq X \times Y = \mu(X) \times \mu(Y)$ holds trivially.

(1.2) We make $\mu(X \times Y) \supseteq \mu(X) \times \mu(Y)$ true, but not the converse. We order all sequences of length 1 or 2 by $<$, and all sequences of length 3 by $\prec$. Suppose $\sigma \in X \times Y - \mu(X \times Y)$. Case 1: $X \times Y$ consists of sequences of length 2. Then, by definition, $\sigma \notin \mu(X) \times \mu(Y)$. Case 2: $X \times Y$ consists of sequences of length 3. Then $\sigma = p\neg q\neg r$, and $\neg p\neg q\neg r \in X \times Y$. So $\{p, \neg p\} \subseteq X$ or $\{p\neg q, \neg p\neg q\} \subseteq X$, but in both cases $\sigma \upharpoonright X \notin \mu(X)$, so $\sigma \notin \mu(X) \times \mu(Y)$. Finally, note that $\mu(TRUE) \not\subseteq \{\neg p\neg q\neg r\} = \mu(\{p, \neg p\}) \times \mu(\{\langle q, r\rangle, \langle q, \neg r\rangle, \langle \neg q, r\rangle, \langle \neg q, \neg r\rangle\})$, so full $(\mu * 1)$ does not hold.

(2) We make interpolation hold, but $\mu(X) \times \mu(Y) \not\subseteq \mu(X \times Y)$: We order all sequences of length 3 by $<$. Shorter sequences are made incomparabel, so for shorter sequences $\mu(X) = X$. Obviously, in general $\mu(X) \times \mu(Y) \not\subseteq \mu(X \times Y)$. But the proof of Proposition 5.3.5 (page 198) goes through as above, only directly, without the use of factorizing and taking μ of the factors.

The reason is that the order on sequences of length 3 behaves in a modular way.

Thus, we can have the result, even if subspaces behave differently, if the completions of the subspace in the whole space behave as they should.

$\square$

5.3.4.2 The extreme interpolants in non-monotonic logics

Two possible ways to solve the problem of interpolation are illustrated in Diagram 5.3.4 (page 201). The main differences between the left and the right hand part, $\mu(\phi) \upharpoonright J'$ vs. $\phi \upharpoonright J'$, are underlined.

Diagram 5.3.4

$$
\begin{tikzcd}[arrows=dash]
 & \phi = (\phi \restriction J \cup J') \times \Pi J'' \arrow[dl] \arrow[ddr] & \\
\mu(\phi) \arrow[d] & & \\
\Pi J \times \underline{\mu(\phi) \restriction J'} \times \Pi J'' \arrow[d] & & \Pi J \times \underline{\phi \restriction J'} \times \Pi J'' \arrow[d] \\
\begin{array}{c} \mu(\Pi J \times \mu(\phi) \restriction J' \times \Pi J'') \\ = \ (?) \\ \mu(\Pi J) \times \mu(\phi) \restriction J' \times \mu(\Pi J'') \end{array} \arrow[dr] & & \begin{array}{c} \mu(\Pi J \times \phi \restriction J' \times \Pi J'') \\ = \ (?) \\ \mu(\Pi J) \times \mu(\phi) \restriction J' \times \mu(\Pi J'') \end{array} \arrow[dl] \\
 & \psi = \Pi J \times (\psi \restriction J' \cup J'') &
\end{tikzcd}
$$

We may take the left path (as done in the proof of Proposition 5.3.5 (page 198)) from top to bottom, apply first μ, then generalize by taking the product, and then apply μ again. Note that we apply μ to a product in the last step. This is all semantical routine (the generalization may be a definability problem when looking at syntax). It depends on multiplication laws whether the outcome is $\mu(\Pi J) \times \mu(\phi) \restriction J' \times \mu(\Pi J'')$, and whether this entails classically ψ.

The path on the right generalizes first, and then applies μ to a product, but it seems more unclear if this has still anything to do with the original $\mu(\phi)$, as ϕ was not a product of 3 components, only of 2 components. This has a similarity to the $\phi \vdash \alpha \mid\sim \psi$ problem discussed in Section 5.3.3 (page 196). We avoided this by taking μ of the original ϕ first on the left hand path. The rest is similar to the left hand path.

We can also consider a variant which is in a sense between the 2 ways discussed here (as the underlined part $\mu(\phi) \upharpoonright J'$ implies the underlined part $\phi \upharpoonright J'$), considering $\Pi J \times (\phi \wedge \psi \upharpoonright J') \times \Pi J''$. And if we look for a most general interpolant, leaving aside considerations about multiplications, we may use the (OR) rule: If $\alpha \upharpoonright J' \mid\sim \psi \upharpoonright J'$, $\alpha' \upharpoonright J' \mid\sim \psi \upharpoonright J'$, then by the (OR) rule, also $(\alpha \vee \alpha') \upharpoonright J' \mid\sim \psi \upharpoonright J'$.

5.3.5 Interpolation for distance based revision

The limiting condition (consistency) imposes a strong restriction: Even for $\phi * TRUE$, the result may need many variables (those in ϕ).

Lemma 5.3.6

Let $|$ satisfy $(| *)$, as defined in Fact 5.2.5 (page 186).

Let $J \subseteq L$, ρ be written in sublanguage J, let ϕ, ψ be written in $L - J$, let ϕ', ψ' be written in $J' \subseteq J$.

Let $(\phi \wedge \phi') * (\psi \wedge \psi') \vdash \rho$, then $\phi' * \psi' \vdash \rho$.

(This is suitable interpolation, but we also need to factorize the revision construction.)

Proof

$(\phi \wedge \phi') * (\psi \wedge \psi') = (\phi * \psi) \wedge (\phi' * \psi')$ by $(| *)$. So $(\phi \wedge \phi') * (\psi \wedge \psi') \vdash \rho$ iff $(\phi * \psi) \wedge (\phi' * \psi') \vdash \rho$, but $(\phi * \psi) \wedge (\phi' * \psi') \vdash \rho$ iff $(\phi' * \psi') \vdash \rho$, as ρ contains no variables of ϕ' or ψ'. $\Box$

5.3.6 The equilibrium logic EQ

5.3.6.1 Introduction and outline

Equilibrium logic, EQ, introduced by Pearce et al., see [PV09], is based on the three valued Goedel logic, based on two worlds, here and there, also called HT logic. It is defined through a model choice function on HT models, and non-monotonic. For motivation and context, the reader is referred to [PV09].

We look at the problem in two different ways. First, we investigate the more classical case where we are interested only in models which have maximal value (i.e., 2), and then look at the more refined situation, as it was described in Section 2.2.2.3 (page 51). Again, we will consider, for both approaches, the three types of interpolation for $\phi \mid\sim \psi : \phi \vdash \alpha \mid\sim \psi$, $\phi \mid\sim \alpha \vdash \psi$, and $\phi \mid\sim \alpha \mid\sim \psi$. We will show that there is not always interpolation of the first two types, but that there is always interpolation of the third type.

In both cases, we have to make a decision: when going from all models to minimal models, which value will we give to those models we eliminate as non-minimal? In the classical case, it was trivial, they get value 0. Now, we could also lower their value by some smaller amount. But we decide to give them value 0, here, too.

5.3.6.2 Basic definition, and definability of chosen models

We first give the basic definition (the model choice function):

Fix a finite propositional language $\mathcal{L}$.

Definition 5.3.1

(The definition is due to [PV09].)

(1) A model m is total iff for no $a \in L$ $m(a) = 1$.

(2) $m \prec m'$ iff

(2.1) for all $a \in L$ $m(a) = 0 \Leftrightarrow m'(a) = 0$ and

(2.2) $\{a \in L : m(a) = 2\} \subset \{a \in L : m'(a) = 2\}$

($\sigma \prec \tau$ iff T is preserved, and H goes down. Thus, only changes from 2 to 1 are possible when $\sigma \prec \tau$.)

(3) $m \in X$ is an equilibrium model of X iff m is total, and there is no $m' \prec m$, $m' \in X$. (We can add $m \prec m$ if $m(x) = 1$ for some $x \in L$, so we can define equilibrium models as minimal HT models for some relation $\prec$.)

(4) $\mu(X)$ will be the "set" of equilibrium models of X.

Recall that, if $f_X : M \to V$ is the model function of X, then $f_{\mu(X)} : M \to V$ will be defined as follows:

$$f_{\mu(X)}(m) := \begin{cases} f_X(m) \quad iff \quad m \quad is \quad chosen \\ 0 \quad\quad\quad\quad otherwise \end{cases}$$

Remark 5.3.7

Consequently, we have a sort of "anti-smoothnes": if a model is not minimal, then any model below it is NOT chosen. Thus, we cannot use general results based on smoothness.

We now show:

Fact 5.3.8

Work in a finite language.

If f_X is definable, then so is $f_{\mu(X)}$.

Proof

Note: We do not need a uniform way to find the defining formula, we just need the formula - even if it is "handcrafted". We will consider all models one by one. Finally, we will AND the definition of f_X with suitable formulae.

If there is no model m such that $f_X(m) > 0$ and $m(p) = 1$ for some p, then $f_{\mu(X)} = f_X$. Consider now all m such that $f_X(m) > 0$ and $m(p) = 1$ for some p, and define ϕ_m for this m. As the language is finite, the set of such m is finite, and if $f_X = f_\phi$, then $f_{\phi \wedge \bigwedge\{\phi_m : \exists p. m(p)=1, f_X(m)>0\}}$ will define $f_{\mu(X)}$. A model is eliminated iff it contains $m(p) = 1$ for some p, or there is some $m' \prec m$ which contains some such p. So, if we can show for m with $m(p) = 1$ for some p $\phi_m(m) = 0$, and $\phi_m(m') = 0$ if $m \prec m'$, and perhaps $\mu_m(m')$ if m' contains some other $m'(p) = 1$, but $\phi_m(m') = 2$ for all other m', then we are done.

Consider now m with $m(p) = 1$ for some p. Let $p_1, \ldots, p_k$ be such that $m(p_i) = 0$, and $q_1, \ldots, q_n$ be such that $m(q_i) > 0$. k may be 0, but not n. Define $\phi_m := \neg\neg p_1 \vee \ldots \vee \neg\neg p_k \vee \neg q_1 \vee \ldots \vee \neg q_n$.

First, consider m. $m(\neg\neg p_i) = 0$ for all i, and also $m(\neg q_i) = 0$ for all i, so $\phi_m(m) = 0$, as desired.

Second, suppose $m \prec m'$. As $m(x) = 0$ iff $m'(x) = 0$, the values for p_i did not change, so still $m'(\neg\neg p_i) = 0$, and the values for q_i may have changed, but not to 0, so still $m'(\neg q_i) = 0$, and $\phi_m(m') = 0$.

Suppose now $m \not\prec m'$. Case 1: $m'(p_i) \neq 0$ for some i, then $m'(\neg\neg p_i) = 2$, and we are done. Case 2: $m'(q_i) = 0$ for some i, then $m'(\neg q_i) = 2$, and we are done. Thus, if $\phi_m(m') = 0$, then $m(x) = 0$ iff $m'(x) = 0$.

So $m(x)$ can only differ from $m'(x)$ on the q_i, and $m'(q_i) \neq 0$. If for all q_i $m'(q_i) = 2$, then $m \prec m'$, and we are done. If not, then $m'(q_i) = 1$ for some i, and it should be eliminated anyway.

□

5.3.6.3 The approach with models of value 2

We first define formally what we want to do:

Definition 5.3.2

(1) Set $M_2(\phi) := \{m \in M : m(\phi) = 2 = TRUE\}$.

(2) Set $\mu_2(\phi) := \{m \in M : m(\phi) = 2$, and m is an equilibrium model $\}$.

(3) Set $\phi \mid\sim \psi$ iff $\mu_2(\phi) \subseteq M_2(\psi)$.

(4) Set $\phi \vdash \psi$ iff $\forall m \in M. m(\phi) \leq m(\psi)$.

We will show that interpolation of type (a) $\phi \vdash \alpha \mid\sim \psi$ and (b) $\phi \mid\sim \alpha \vdash \psi$ may fail, but interpolation of type (c) $\phi \mid\sim \alpha \mid\sim \psi$ will exist.

Definability of the interpolant will be shown using the definability results for HT. We can use the techniques and results developped there ("neglecting" some variables), and see that the semantical interpolant is definable, so we have also syntactical interpolation.

Example 5.3.3

(EQ has no interpolation of the form $\phi \vdash \alpha \mid\sim \psi$.)

Work with 3 variables, a, b, c.

Consider $\Sigma := \{\langle 0, 2, 2\rangle, \langle 2, 1, 0\rangle, \langle 2, 2, 0\rangle\}$.

By the above, and classical behaviour of "or" and "and", Σ is definable by $\phi := (\neg a \wedge b \wedge c) \vee (a \wedge \neg\neg b \wedge \neg c)$, i.e. $\Sigma = \{m : m(\phi) = 2\}$.

Note that $\langle 2,2,0\rangle$ is total, but $\langle 2,1,0\rangle \prec \langle 2,2,0\rangle$, thus $\mu(\Sigma) = \{\langle 0,2,2\rangle\}$.

So $\Sigma \mid\sim c = 2$ (or $\Sigma \mid\sim \Box c$). Let $X' := \{a,b\}$, $X'' := \{c\}$.

All possible interpolants Γ must not contain a or b as essential variables, and they must contain Σ. The smallest candidate Γ is $\Pi X' \times \{0,2\}$. But $\sigma := \langle 0,0,0\rangle \in \Gamma$, σ is total, and there cannot be any $\tau \prec \sigma$, so $\sigma \in \mu(\Gamma)$, so $\Gamma \not\mid\sim c = 2$.

For completeness' sake, we write all elements of Γ :

$\langle 0,0,0\rangle$ $\langle 0,0,2\rangle$

$\langle 0,1,0\rangle$ $\langle 0,1,2\rangle$

$\langle 0,2,0\rangle$ $\langle 0,2,2\rangle$

$\langle 1,0,0\rangle$ $\langle 1,0,2\rangle$

$\langle 1,1,0\rangle$ $\langle 1,1,2\rangle$

$\langle 1,2,0\rangle$ $\langle 1,2,2\rangle$

$\langle 2,0,0\rangle$ $\langle 2,0,2\rangle$

$\langle 2,1,0\rangle$ $\langle 2,1,2\rangle$

$\langle 2,2,0\rangle$ $\langle 2,2,2\rangle$

Recall that no sequence containing 1 is total, and when we go from 2 to 1, we have a smaller model. Thus, $\mu(\Gamma) = \{\langle 0,0,0\rangle, \langle 0,0,2\rangle\}$.

Example 5.3.4

(EQ has no interpolation of the form $\phi \mid\sim \alpha \vdash \psi$.)

Consider 2 variables, a, b, and $\Sigma := \{0,2\} \times \{0,1,2\}$

No σ containing 1 can be in $\mu(\Sigma)$, as a matter of fact, $\mu(\Sigma) = \{\langle 0,0\rangle, \langle 2,0\rangle\}$. Σ is defined by $a \vee \neg a$, $\mu(\Sigma)$ is defined by $(a \vee \neg a) \wedge \neg b$.

So we have $a \vee \neg a \mid\sim b \vee \neg b$, even $a \vee \neg a \mid\sim \neg b$.

The only possible interpolants are TRUE or FALSE. $a \vee \neg a \not\mid\sim FALSE$, and $TRUE \not\vdash \neg b$.

Fact 5.3.9

EQ has interpolation of the form $\phi \mid\sim \alpha \mid\sim \psi$.

Proof

Let $\phi \mid\sim \psi$, i.e., $\mu_2(\phi) \subseteq M_2(\psi)$. We have to find α such that $\mu_2(\phi) \subseteq M_2(\alpha)$, and $\mu_2(\alpha) \subseteq M_2(\psi)$.

Let $J = I(\phi)$, $J'' = I(\psi)$. Consider $X := \Pi J \times (\mu_2(\phi) \restriction J') \times \Pi J''$. By the same arguments ("neglecting" J and J''), X is definable as $M_2(\alpha)$ for some α.

Obviously, $\mu_2(\phi) \subseteq M_2(\alpha)$. Consider now $\mu_2(\alpha)$, we have to show $\mu_2(\alpha) \subseteq M_2(\psi)$. If $\mu_2(\alpha) = \emptyset$, we are done, so suppose there is $m \in \mu_2(\alpha)$. Suppose $m \notin M_2(\psi)$. There is $m' \in \mu_2(\phi)$, $m' \restriction J' = m \restriction J'$.

We use now + for concatenation.

Consider $m'' = (m \upharpoonright J) + m' \upharpoonright (J' \cup J'')$. As $m' \in \mu_2(\phi) \subseteq M_2(\phi)$, and $M_2(\phi) = \Pi J \times M_2(\phi) \upharpoonright (J' \cup J'')$, $m'' \in M_2(\phi)$. $m \upharpoonright (J \cup J') = m'' \upharpoonright (J \cup J')$, thus by $J'' \subseteq I(\psi)$, $m'' \notin M_2(\psi)$. Thus, $m'' \notin \mu_2(\phi)$. So either there is $n \in M_2(\phi)$ such that $n(y) = 0$ iff $m''(y) = 0$ and $\{y : n(y) = 2\} \subset \{y : m''(y) = 2\}$ or $m''(y) = 1$ for some $y \in L$. Suppose $m''(y) = 1$ for some y. y cannot be in $J' \cup J''$, as $m'' \upharpoonright (J' \cup J'') = m' \upharpoonright (J' \cup J'')$, and $m' \in \mu_2(\phi)$. y cannot be in J, as $m'' \upharpoonright J = m \upharpoonright J$, and $m \in \mu_2(X)$.

So there must be $n \in M_2(\phi)$ as above. Case 1: $\{y \in J' \cup J'' : n(y) = 2\} \subset \{y \in J' \cup J'' : m''(y) = 2\}$. Then $n' = m' \upharpoonright J + n \upharpoonright (J' \cup J'')$ would eliminate m' from $\mu_2(\phi)$, so this cannot be. Thus, $n \upharpoonright (J' \cup J'') = m'' \upharpoonright (J' \cup J'')$. So $\{y \in J : n(y) = 2\} \subset \{y \in J : m''(y) = 2\} = \{y \in J : m(y) = 2\}$ by $m'' \upharpoonright J = m \upharpoonright J$. Consider now $n' = n \upharpoonright J + m \upharpoonright (J' \cup J'')$. $n' \in \Pi J \times \mu_2(\phi) \upharpoonright J' \times \Pi J''$. $n'(y) = 0$ iff $m(y) = 0$ by construction of n' and n. So $n' \prec m$, and $m \notin \mu_2(\Pi J \times (\mu_2(\phi) \upharpoonright J') \times \Pi J'')$, contradiction.

□

5.3.6.4 The refined approach

We consider now more truth values, in the sense that $\phi \mid\sim \psi$ iff $f_{\mu(f_\phi)} \leq f_\psi$ - and not only restricted to value 2, as in Definition 5.3.2 (page 205). The arguments and examples will be the same, they are given for completess' sake only.

Again, we show that there need not be interpolation of the forms $\phi \vdash \alpha \mid\sim \psi$ or $\phi \mid\sim \alpha \vdash \psi$, but there will be interpolation of the type $\phi \mid\sim \alpha \mid\sim \psi$.

Example 5.3.5

(EQ has no interpolation of the form $\phi \vdash \alpha \mid\sim \psi$.)

Work with 3 variables, a, b, c. Models will be written as $\langle 0,0,0 \rangle$, etc., in the obvious meaning.

Consider $\phi := (\neg a \wedge b \wedge c) \vee (a \wedge \neg\neg b \wedge \neg c)$.

For $\langle 0,1,1 \rangle$, $\langle 0,1,2 \rangle$, $\langle 0,2,1 \rangle$, $\langle 1,1,0 \rangle$, $\langle 1,2,0 \rangle$, f_ϕ has value 1, for $\langle 0,2,2 \rangle$, $\langle 2,1,0 \rangle$, $\langle 2,2,0 \rangle$ f_ϕ has value 2, all other values are 0. The only chosen model is $\langle 0,2,2 \rangle$, all others contain 1, or are minimized. So $f_{\mu(\phi)}$ has value 2 for $\langle 0,2,2 \rangle$, all other values are 0. Obviously, $f_{\mu(\phi)} \leq f_c$, so $\phi \mid\sim c$.

As shown in Fact 4.4.5 (page 150), we can define with c only c, $\neg c$, $\neg\neg c$, $c \to c$, $\neg(c \to c)$, $\neg\neg c \to c$ (up to semantical equivalence). But none is an interpolant of the type $\phi \vdash \alpha \mid\sim \psi$: The left hand condition fails for c, $\neg c$, $\neg\neg c$, $\neg(c \to c)$, the right hand condition fails for $c \to c$ and $\neg\neg c \to c$, as $f_{\mu(c \to c)}(\langle 0,0,0 \rangle) = f_{\mu(\neg\neg c \to c)}(\langle 0,0,0 \rangle) = 2$.

Example 5.3.6

(EQ has no interpolation of the form $\phi \mid\sim \alpha \vdash \psi$.)

Consider 2 variables, a, b, $\phi := a \vee \neg a$.

$f_\phi(m) = 1$ iff $m(a) = 1$, and 2 otherwise. Note that (the model) $\langle 1,0 \rangle \prec \langle 2,0 \rangle$, but $f_\phi(\langle 1,0 \rangle) = 1$, $f_\phi(\langle 2,0 \rangle) = 2$, so this minimization does not "count". Consequently, $f_{\mu(\phi)}(m) = 2$ iff $m = \langle 0,0 \rangle$ or $m = \langle 2,0 \rangle$, and $f_{\mu(\phi)}(m) = 0$ otherwise. Thus, $\phi \mid\sim \neg b$. But $\phi \not\mid\sim FALSE$, and $TRUE \not\vdash \neg b$.

□

Fact 5.3.10

EQ has interpolation of the form $\phi \mid\sim \alpha \mid\sim \psi$.

Proof

See Diagram 5.3.3 (page 199) for illustration.

Let $L = J \cup J' \cup J''$, $J'' = I(\phi)$, $J = I(\psi)$. As ϕ does not contain any variables in J'' in an essential way, $f_\phi(m) = f_\phi(m')$ if $m \upharpoonright J \cup J' = m' \upharpoonright J \cup J'$. Thus, if $a \in J''$, and $m \upharpoonright L - \{a\} = m' \upharpoonright L - \{a\} = m'' \upharpoonright L - \{a\}$, and $m(a) = 0$, $m'(a) = 1$, $m''(a) = 2$, then by $f_\phi(m) = f_\phi(m') = f_\phi(m'')$, neither m' nor m'' survives minimization, i.e., $f_{\mu(\phi)}(m') = f_{\mu(\phi)}(m'') = 0$. Thus, if $m(a) \neq 0$ for some $a \in J''$, then $f_{\mu(\phi)}(m) = 0$. On the other hand, if $m \upharpoonright J' \cup J'' = m' \upharpoonright J' \cup J''$, then $f_\psi(m) = f_\psi(m')$.

Define now the semantic interpolant by $h(m) := sup\{f_{\mu(\phi)}(m') : m' \upharpoonright J' = m \upharpoonright J'\}$. Obviously, $f_{\mu(\phi)} \leq h$, so, if $h = f_\alpha$ for some α, then $\phi \mid\sim \alpha$. It remains to show that $f_{\mu(h)} \leq f_\psi$, then $\alpha \mid\sim \psi$, and we are done.

For the same reasons as discussed above, $f_{\mu(h)}(m) = 0$ if $m(a) \neq 0$ for some $a \in J \cup J''$. Take now arbitrary m, we have to show $f_{\mu(h)}(m) \leq f_\psi(m)$. If $m(a) \neq 0$ for some $a \in J \cup J''$, there is nothing to show. So suppose $m(a) = 0$ for all $a \in J \cup J''$. By the above, $h(m) = sup\{f_{\mu(\phi)}(m')$: $m' \upharpoonright J' = m \upharpoonright J' \wedge \forall a \in J''.m'(a) = 0\}$, so, as $m(a) = 0$ for $a \in J''$, $h(m) = sup\{f_{\mu(\phi)}(m')$: $m' \upharpoonright J' \cup J'' = m \upharpoonright J' \cup J''\}$. By prerequisite, $f_{\mu(\phi)}(m') \leq f_\psi(m')$ for all m', but ψ does not contain essential variables in J, so if $m' \upharpoonright J' \cup J'' = m \upharpoonright J' \cup J''$, then $f_\psi(m) = f_\psi(m')$, thus $h(m) \leq f_\psi(m)$, but $f_{\mu(h)} \leq h$, so $f_{\mu(h)} \leq h(m) \leq f_\psi(m)$.

□

5.4 Context and structure

The discussion in this Section is intended to open the perspective and separate support from attack, and, even more broadly, separate logic from manipulation of model sets. But this is not pursued here, and intended to be looked at in future research.

We take the importance of condition $(\mu * 3)$ (or $(S * 3)$) as occasion for a broader remark.

(1) This condition points to a weakening of the Hamming condition:

Adding new "branches" in X' will not give new minimal elements in X'', but may destroy other minimal elements in X''. This can be achieved by a sort of semi-rankedness: If ρ and σ are different only in the $X'-$part, then $\tau \prec \rho$ iff $\tau \prec \sigma$, but not necessarily $\rho \prec \tau$ iff $\sigma \prec \tau$.

(2) In more abstract terms:

When we separate support from attack (support: a branch σ' in X' supports a continuation σ'' in X'' iff $\sigma \circ \sigma''$ is minimal, i.e. not attacked, attack: a branch τ in X' attacks a continuation σ'' in X'' iff it prevents all $\sigma \circ \sigma''$ to be minimal), we see that new branches will not support any new continuations, but may well attack continuations.

More radically, we can consider paths σ'' as positive information, σ' as potentialy negative information. Thus, Π' gives maximal negative information, and thus smallest set of accepted models.

(3) We can interpret this as follows: X'' determines the base set. X' is the context. This determines the choice (subset of the base set). We compare to preferential structures: In preferential structures, $\prec$ is not part of the language either, it is context. And we have the same behaviour as shown in the fundamental property of preferential structures: the bigger the set, the more attacks are possible.

(4) The concept of size looks only at the result of support and attack, so it is necessarily somewhat coarse. Future research should also investigate both concepts separately.

We broaden this.

Following a tradition begun by Kripke, one has added structure to the set of classical models, reachability, preference, etc. Perhaps one should emphasize a more abstract approach, as done by one the authors e.g. in [Sch92], and elaborated in [Sch04], see in particular the distinction between structural and algebraic semantics in the latter. Our suggestion is to separate structure from logic in the semantics, and to treat what we called context above by a separate "machinery". So, given a set X of models, we have some abstract function f, which chooses the models where the consequences hold, $f(X)$.

Now, we can put into this "machinery" whatever we want.

The abstract properties of preferential or modal structures are well known.

But we can also investigate non-static f, where f changes in function of what we already did - "reacting" to the past.

We can look at usual properties of f, complexity, generation by some simple structure like a special machine, etc.

So we advocate the separation of usual, classical semantics, from the additional properties, which are treated "outside". It might be interesting to forget altogether about logic, classify those functions or more complicated devices which correspond to some logical property, and investigate them and their properties.

5.5 Interpolation for argumentation

Arguments (e.g., in inheritance), are sometimes ordered by a partial order only. We may define: ψ follows from ϕ in argumentation iff for every argument for ϕ there is a better or equal argument for ψ. It is not sufficient to give just one argument, there might not be a best one. We have to consider the *set* of all arguments.

Consequently, if V = truth value set = set of arguments with a partial order $\preceq$, we have to look at functions $f : M \to \mathcal{P}(V)$, where to each model m (M the model set) is assigned a set of arguments (which support that "m belongs to f".) Example: is m a weevil? Yes, it has a long nose. Yes, it has articulate antennae .... Thus, $f_{weevil}(m) = \{$ long nose, articulate antennae $\}$. We have to define $\leq$ on $\mathcal{P}(V)$. We think a good definition is:

Definition 5.5.1

For $A, B \subseteq V$ (V with partial order $\preceq$) we define:

$A \leq B$ iff $\forall a \in A \exists b \in B.a \preceq b$.

This seems to be a decent definition of comparison of argument sets. Why not conversely? Suppose we have a very shaky argument b for ψ, then to say that arguments for ψ are better than arguments for ϕ, we would need an even worse argument for ϕ. This does not seem right.

Thus, we define for arbitrary model functions f and g :

Definition 5.5.2

Let $f, g : M \to \mathcal{P}(V)$. We say f entails g iff:

$f \leq g$ iff $\forall a \in f(m) \exists b \in g(m).a \preceq b$.

In total orders, sup and inf were defined. We want for sup: $A, B \leq sup(A, B)$, and $A, B \leq C \Rightarrow sup(A, B) \leq C$. So we define:

Definition 5.5.3

For a set $\mathcal{A}$ of argument sets, define $sup(\mathcal{A}) := \bigcup \mathcal{A}$.

Fact 5.5.1

We have

(1) For all $A \in \mathcal{A}$ $A \leq sup(\mathcal{A})$.

(2) If for all $A \in \mathcal{A}$ $A \leq B$, then $sup(\mathcal{A}) \leq B$.

Proof

Trivial by definition of $\leq$. $\square$.

What is the inf? A definition should also work if the order is empty. Then, $inf(A, B) = A \cap B$, which may be empty. This is probably not what we want. It is probably best to leave inf undefined.

But we can replace $A \leq inf(B, C)$ by $A \leq B$ and $A \leq C$, so we can work with inf on the right of $\leq$ without a definition, replacing it by the universal quantifier (or, equivalently, by AND).

For interpolation, for $L = J \cup J' \cup J''$, f insensitive to J, g insensitive to J'', $f(m) \leq g(m)$ for all $m \in M$, we looked at $f^+(m_{J'}) := sup\{f(m') : m \restriction J' = m' \restriction J'\}$ and $g^-(m_{J'}) := inf\{g(m') : m \restriction J' = m' \restriction J'\}$. We showed that $f^+(m_{J'}) \leq g^-(m_{J'})$.

We have to modify and show:

Fact 5.5.2

$sup\{f(m') : m \restriction J' = m' \restriction J'\} := \bigcup\{f(m') : m \restriction J' = m' \restriction J'\} \leq g(m'')$ for all m'' such that $m \restriction J' = m'' \restriction J'$.

Proof

By definition of $\leq$, it suffices to show that $\forall m' \forall m'' (m \upharpoonright J' = m' \upharpoonright J' \wedge m \upharpoonright J' = m'' \upharpoonright J' \Rightarrow f(m') \leq g(m''))$.

Take m' and m'' as above, so $m' \upharpoonright J' = m \upharpoonright J' = m'' \upharpoonright J'$. Define m_0 by $m_0 \upharpoonright J = m'' \upharpoonright J$, $m_0 \upharpoonright J' \cup J'' = m' \upharpoonright J' \cup J''$. As f is insensitive to J, $f(m') = f(m_0) \leq g(m_0)$ by prerequisite. Note that $m_0 \upharpoonright J \cup J' = m'' \upharpoonright J \cup J'$, as $m_0 \upharpoonright J' = m' \upharpoonright J' = m'' \upharpoonright J'$. As g is insensitive to J'', $g(m_0) = g(m'')$. So we have $f(m') = f(m_0) \leq g(m_0) = g(m'')$.

□

Fact 5.5.3

$f^+(m_{J'})$ is an interpolant for f and g under above prerequisites.

Proof

Define $h(m) := f^+(m_{J'})$. We have to show that h is an interpolant. $f(m) \leq h(m)$ is trivial by definition. It remains to show that $h(m) \leq g(m)$ for all m. $h(m) := sup\{f(m') : m \upharpoonright J' = m' \upharpoonright J'\}$ $\leq g(m)$ iff $\forall m'(m' \upharpoonright J' = m \upharpoonright J' \Rightarrow f(m') \leq g(m))$, but this is a special case of the proof of Fact 5.5.2 (page 210). □

Note that the same approach may also be used in other contexts, e.g. considering worlds in Kripke structures as truth values, $w \in M(\phi)$ iff $w \in f_\phi$. All we really need is some kind of sup and inf.

Chapter 6

Neighbourhood semantics

6.1 Introduction

Neighbourhood semantics, probably first introduced by D.Scott and R.Montague in [Sco70] and [Mon70], and already used for deontic logic by O.Pacheco in [Pac07] to avoid unwanted weakening of obligations, seem to be useful for many logics:

(1) in preferential logics, they describe the limit variant, where we consider neighbourhoods of an ideal, usually inexistent, situation,

(2) in approximative reasoning, they describe the approximations to the final result,

(3) in deontic and default logic, they describe the "good" situations, i.e., deontically acceptable, or where defaults have fired.

Neighbourhood semantics are used, when the "ideal" situation does not exist (e.g., preferential systems without minimal elements), or are too difficult to obtain (e.g., "perfect" deontic states).

6.1.1 Defining neighbourhoods

Neighbourhoods can be defined in various ways:

- by algebraic systems, like unions of intersections of certain sets (but not complements),
- quality relations, which say that some points are better than others, carrying over to sets of points,
- distance relations, which measure the distance to the perhaps inexistant ideal points.

The relations and distances may be given already by the underlying structure, e.g., in preferential structures, or they can be defined in a natural way, e.g., from a systems of sets, as in deontic logic or default logic. In these cases, we can define a distance between two points by the number or set of deontic requirements or default rules which one satisfies, but not the other. A quality relation is defined in a similar way: a point is better, if it satisfies more requirements or rules.

6.1.2 Additional requirements

With these tools, we can define properties neighbourhoods should have. E.g., we may require them to be downward closed, i.e., if $x \in N$, where N is a neighbourhood, $y \prec x$, y is better than x, then y should also be in N. This is a property we will certainly require in neighbourhood semantics for preferential structures (in the limit version). For these structures, we will also require that for every $x \notin N$, there should be some $y \in N$ with $y \prec x$. We may also require that, if $x \in N$, $y \notin N$, and y is in some aspect better than x, then there must be $z \in N$, which is better than both, so we have some kind of "ceteris paribus" improvement.

6.1.3 Connections between the various properties

There is a multitude of possible definitions (via distances, relations, set systems), and properties, so it is not surprising that one can investigate a multitude of connections between the different possible definitions of neighbourhoods. We cannot cover all possible connections, so we compare only a few cases, and the reader is invited to complete the picture for the cases which interest him. The connections we examined are presented in Section 6.3.4 (page 225).

6.1.4 Various uses of neighbourhood semantics

We also distinguish the different uses of the systems of sets thus characterized as neighbourhoods: we can look at all formulas which hold in (all or some) such sets (as in neighbourhood semantics for preferential logics), or at the formulas which exactly describe them. The latter reading avoids the infamous Ross paradox of deontic logic. This distinction is simple, but basic, and did probably not receive the attention it deserves, in the literature.

6.2 Detailed overview

Our starting point was to give the "derivation" in deontic systems a precise semantical meaning. We extend this now to encompass the following situations:

(1) Deontic systems, including contrary-to-duty obligations

(2) Default systems a la Reiter.

(3) The limit version of preferential structures

(4) Approximative logic

We borrow the word "neighbourhood" from analysis and topology, but should be aware that our use will be, at least partly, different.

Common to topology and our domain is that - for reasons to be discussed - we are not only interested in one ideal point or one ideal set, but in sets which are in some sense bigger, and whose elements are in some sense close to the "ideal".

6.2.1 Motivation

What are the reasons to consider some kind of "approximation"?

(1) First, the "ideal" might not exist, e.g.:

(1.1) In preferential structures, minimal models are the ideal, but there might be none, due to infinite descending chains. So the usual approach via minimal models leads to inconsistency, we have to take the limit approach, see Definition 2.3.5 (page 62). The same holds, e.g., for theory revision or counterfactual conditionals without closest worlds.

(1.2) Default rules might be contradictory, so the ideal (all defaults are satisfied) is impossible to obtain.

(2) Second, the ideal might exist, but be too difficult to obtain, e.g.:

(2.1) In deontic logic, the requirements to lead a perfectly moral life might just be beyond human power. The same may hold for other imperative systems. E.g., we might be obliged to post the letter and to water the plants, but we have not time for both, so doing one or the other is certainly better than nothing (so the "or" in the Ross paradox is *not* the problem).

(2.2) It might be too costly to obtain perfect cleanliness, so we have to settle with sufficiently clean.

(2.3) Approximate reasoning will try to find better and better answers, perhaps without hope to find an ideal answer,

(3) Things might be even more complicated by a (partial or total) hierarchy between aims. E.g., it is a "stronger" law not to kill than not to steal.

6.2.2 Tools to define neighbourhoods

To define a suitable notion of neighbourhood, we may have various tools:

(1) We may have a quality relation between points, where $a \prec b$ says that b is in some sense "better" than a.

Such relations are, e.g., given in:

(1.1) in preferential structures, it is the preference relation

(1.2) in defaults, a (normal) default gives a quality relation: the situations which satisfy it, are "better" than those which do not - here, a default gives a quality relation not only to two situations, but usually between two *sets* of situations - the same holds for deontic logics and other imperative systems,

(1.3) We may have borders separating subsets, with a direction, where it is "better" inside (or outside) the border, as in $X-$logic, see [BS85], and [Sch04].

(1.4) in approximation, one situation might be closer than the other to the ideal.

(2) We may have several such relations, which may also partly contradict each other, and we may have a relation of importance between different $\prec$ and $\prec'$, (as in the example of not to kill or not to steal). The better a situation is, the closer it should be to our ideal.

(3) We may have a distance relation between points, and these distances may be partially or totally ordered. With the distance relation, we can measure the distance of a point to the ideal points (all of them, the closest one, the most distant ideal point, etc.). Even if the ideal points do not exist, we can perhaps find a reasonable measure of distance to them.

This can be found in distance semantics for theory revision and counterfactuals. There, it is the "closeness" relation, we are interested only in the closest models, and if they do not exist, in the sufficiently close ones (the limit approach),

6.2.3 Additional requirements

But there might still be other requirements:

(1) We might postulate that neighbourhoods do not only contain all sufficiently good points, but also do *not* contain any points which are too bad.

(2) We may require that they are closed under certain operations, e.g.:

(2.1) If x is in a neighbourhood X, and y better than x, then it should also be in X, (closure under improvement, see Definition 6.3.10 (page 223)).

(2.2) For all y and any neighbourhood X, there should be some $x \in X$, which is better than y.
(This and the preceeding requirement are those of MISE, see Definition 2.3.5 (page 62).)

(2.3) If we have a notion of distance, and x, x' are in a neighbourhood X, then anything "between" x and x' should be in X (X is convex). Thus, when we move in X, we do not risk to leave X.

(2.4) Similarly, if $x \in X$, and y is an ideal point, then everything between x and y is in X. Or, if $x \in X$, and y is an ideal point closest to x, then everything between x and y should be in X. So, when we improve our situation, we will not leave the neighbourhood. (A star shaped set around an ideal point may satisfy this requirement, without being convex.) (See Definition 6.3.12 (page 224).)

(2.5) If we have to satisfy several requirements, we can ask whether this is possible independently for those requirements, or if we have to sacrify one requirement in order to satisfy another. If the latter is the case, is there a hierarchy of requirements?

(2.6) Elements in a "good" neighbourhood should be better than the others:
If $x \in X$ and the closest (to x) $y \in \boldsymbol{C}(X)$, then $x \prec y$ should hold, and, conversely, if $y \in \boldsymbol{C}(X)$, and the closest (to y) $x \in X$, then $x \prec y$ should hold, see Definition 6.3.11 (page 224),

(2.7) It might be desirable to improve quality by moving into a good neighbourhood, without sacrificing anything achieved already, this is supposed to capture the "ceteris paribus" idea:
If $x \in X$ and $y \notin X$ satisfy a set R of rules, then there is $x' \in X$, which also satisfies R, and which is better than y, see Definition 6.3.5 (page 220).

(2.8) Given a set of "good" sets, we might be able to construct all good neighbourhoods by simple algebraic operations: Any good neighbourhood X is a union of intersections of the "good" sets, see Definition 6.3.2 (page 219),

(2.9) Finally, if this exists, the set of ideal points should probably satisfy our criteria, the set of ideal points should be a "good" neighbourhood.

Of particular interest are requirements which are in some sense independent:

(1) We should try to satisfy an obligation, without violating another obligation, which was not violated before.

(2) The idea behind the Stalnaker/Lewis semantics for counterfactuals, (see [Sta68], [Lew73]), is to look at the closest, i.e., minimally changed situations. "If it were to rain, I would use an umbrella" means something like: "If it were to rain, and there were not a very strong wind" (there is no such wind now), "if I had an umbrella" (I have one now), etc., i.e. if things were mostly as they are now, with the exception that now it does not rain, and in the situation I speak about it rains, then I will use an umbrella.

(3) The distance semantics for theory revision looks also (though with a slightly different formal approach) at the closest, minimally changed, situations.

(4) This idea of "ceteris paribus" is the attempt to isolate a necessary change from the rest of the situation, and is thus intimately related to the concept of independence. Of course, a minimal change might not be possible, but small enough changes might do. Consider, e.g., the constant function $f : [0,1] \to [0,1]$, $f(x) := 0$, and we look for a minimally changed *continous* function with $f(0.5) := 1$. This does not exist. So we have to do with approximation, and look at functions "sufficiently" close to the first function. This is one of the reasons we have to look at the limit variant of theory revision and counterfactuals.

Remark: We do not look here at *paths* which lead (efficiently?) to better and better situations.

6.2.4 Interpretation of the neighbourhoods

Once we have identified our "good" neighbourhoods, we can interpret the result in several ways:

(1) We can ask what holds in *all* good neighbourhoods.

(2) We can ask what holds (finally) in *some* good neighbourhood - this is the approach for limit preferential structures and similar situations.

(3) We may be not so much interested in what holds in all or some good neighbourhoods, but to describe them: This is the problem of the semantics of a system of obligations. In short: what distinguishes a good from a bad set of situations.

Such characterization of "good" situations will give us a new semantics not only for deontic logics, and thus a precise semantical meaning for the "derivation" in deontic systems, see below for a justification, but also for defaults, preferential structures, etc. In particular, such descriptions will not necessarily be closed under arbitrary classical weakening - see the infamous Ross Paradox, Example 6.4.1 (page 233).

6.2.5 Overview of the different lines of reasoning

This chapter is conceptually somewhat complicated, therefore we give now an overview of the different aspects:

(1) We look at different tools and ways to define neighbourhoods, using distances, quality relations, and perhaps combining them, or purely algebraic ways like unions, intersections, etc.

(2) We look at additional requirements for neighbourhoods, using such tools, like closure principles.

(3) We investigate how to obtain such natural relations, distances, etc. from different structures, e.g., from obligations, defaults, preferential models, etc.

(4) We look at various possibilities to interpret the neighbourhood systems we have constructed, e.g., we can ask what holds in all or some neighbourhoods, what finally holds in neighbourhoods (when we have a grading of the neighbourhoods), or what characterizes the neighbourhoods, e.g., in the case of deontic logic.

(5) We conclude (unsystematically) with connections between the different concepts.

6.2.6 Extensions

This might be the place to make a remark on extensions.

An extension is, roughly, a maximal consistent set of information, or, a smallest non-empty set of models. In default logic, we can follow contradictory default to the end of reasoning, and obtain perhaps vcontradictory information, likewise in inheritance nets, etc. Usually, one then takes the intersection of extensions, what is true in all extensions, which is - provided the laguage is adequate - the "OR" of the extensions.

But we can also see preferential structures as resulting in extensions, where every minimal model is an extension:

Consider a preferential structure with 4 models, say pq, $p\neg q$, $\neg pq$, $\neg p\neg q$, ordered by $pq \prec p\neg q$, $\neg pq \prec p\neg q$. Then we can see the relation roughly as two defaults: $p\neg q : pq$, and $p\neg q : \neg pq$, with two extensions: pq and $\neg pq$. So, we can see a preferential structure as having usually many extensions (unless there is a single best model, of course), and we take as result the intersection of extensions, i.e., the theory which holds in *all* minimal models.

In preferential structures, the construction of the set of minimal models is a one-step process: a model is in or out. In defaults, for instance, the construction is more complicated, we branch in the process. This is what may make the construction problematic, and gives rise to different approaches like taking immediately the intersection of extensions in inheritance networks, etc. But this difference to preferential structures is in the *process* of construction, it is not in the outcome.

These questions are intimately related to our neighbourhood semantics, as the constructions can be seen as an approximation to the ideal, the final outcome.

6.3 Tools and requirements for neighbourhoods and how to obtain them

6.3.1 Tools to define neighbourhoods

Background

We often work with an additional structure, some $\mathcal{O} \subseteq \mathcal{P}(U)$, where U is the universe (intuitively, U is a set of propositional models), which allows to define distances and quality relations in a natural way. Intuitively, $\mathcal{O}$ is a base set of "good" sets, from which we will construct other "good" sets.

Basically, x is better than y, iff x is in more (as a set or by counting) $O \in \mathcal{O}$ than y is, and the distance between x and y is the set (or cardinality) of $O \in \mathcal{O}$ where $x \in O$, $y \notin O$, or vice versa.

Sometimes, it is more appropriate to work with sequences of $0/1$, where 1 stands for O, 0 for $\boldsymbol{C}(O)$ for $O \in \mathcal{O}$.

Thus, we work with sets Σ of sequences. Note that Σ need not contain all possible sequences, corresponding to the possibility that, e.g., $O \cap O' = \emptyset$ for $O, O' \in \mathcal{O}$.

Moreover, we may have a difference in quality between O and $\boldsymbol{C}(O)$: if O is an obligation, then $x \in O$ is - at least for this obligation - better than $x' \notin O$. The same holds for defaults of the type $: \phi/\phi$, with $O = M(\phi)$. We will follow the tradition of preferential structures, and "smaller" will mean "better".

6.3.1.1 Algebraic tools

Let here again $\mathcal{O} \subseteq \mathcal{P}(U)$.

Definition 6.3.1

Given a finite propositional laguage $\mathcal{L}$ defined by the set $v(\mathcal{L})$ of propositional variables, let $\mathcal{L}_\wedge$ be the set of all consistent conjunctions of elements from $v(\mathcal{L})$ or their negations. Thus, $p \wedge \neg q \in \mathcal{L}_\wedge$ if $p, q \in v(\mathcal{L})$, but $p \vee q$, $\neg(p \wedge q) \notin \mathcal{L}_\wedge$. Finally, let $\mathcal{L}_{\vee\wedge}$ be the set of all (finite) disjunctions of formulas from $\mathcal{L}_\wedge$. (As we will later not consider all formulas from $\mathcal{L}_\wedge$, this will be a real restriction.)

Given a set of models M for a finite language $\mathcal{L}$, define $\phi_M := \bigwedge\{p \in v(\mathcal{L}) : \forall m \in M.m(p) = v\} \wedge \bigwedge\{\neg p : p \in v(\mathcal{L}), \forall m \in M.m(p) = f\} \in \mathcal{L}_\wedge$. (If there are no such p, set $\phi_M := TRUE$.)

This is the strongest $\phi \in \mathcal{L}_\wedge$ which holds in M.

Definition 6.3.2

$X \subseteq U'$ is (ui) (for union of intersections) iff there is a family $\mathcal{O}_i \subseteq \mathcal{O}$, $i \in I$ such that $X = (\bigcup\{\bigcap \mathcal{O}_i : i \in I\}) \cap U'$.

Unfortunately, as we will see later, this definition is not very useful for simple relativization.

Definition 6.3.3

Let $\mathcal{O}' \subseteq \mathcal{O}$. Define for $m \in U$ and $\delta : \mathcal{O}' \to 2 = \{0, 1\}$

$m \models \delta :\Leftrightarrow \forall O \in \mathcal{O}'(m \in O \Leftrightarrow \delta(O) = 1)$

Definition 6.3.4

$\mathcal{O}$ is independent iff $\forall \delta : \mathcal{O} \to 2.\exists m \in U.m \models \delta$.

Obviously, independence does not inherit downward to subsets of U.

Definition 6.3.5

$\mathcal{D}(\mathcal{O}) := \{X \subseteq U' : \forall \mathcal{O}' \subseteq \mathcal{O}\ \forall \delta : \mathcal{O}' \to 2$

$((\exists m, m' \in U,\ m, m' \models \delta,\ m \in X, m' \notin X) \Rightarrow (\exists m'' \in X.m'' \models \delta \wedge m'' \prec_s m'))\}$

This property expresses that we can satisfy obligations independently: If we respect O, we can, in addition, respect O', and if we are hopeless kleptomaniacs, we may still not be a murderer. If $X \in \mathcal{D}(\mathcal{O})$, we can go from $U - X$ into X by improving on all $O \in \mathcal{O}$, which we have not fixed by δ, if δ is not too rigid.

6.3.1.2 Relations

We may have an abstract relation $\preceq$ of quality on the domain, but we may also define it from the structure of the sequences, as we will do now.

Definition 6.3.6

Consider the case of sequences.

Given a relation $\preceq$ (of quality) on the codomain, we extend this to sequences in Σ :

$x \sim y :\Leftrightarrow \forall i \in I(x(i) \sim y(i))$

$x \preceq y :\Leftrightarrow \forall i \in I(x(i) \preceq y(i))$

$x \prec y :\Leftrightarrow \forall i \in I(x(i) \preceq y(i))$ and $\exists i \in I(x(i) \prec y(i))$

In the $\in -$case, we will consider $x \in i$ better than $x \notin i$. As we have only two values, true/false, it is easy to count the positive and negative cases (in more complicated situations, we might be able to multiply), so we have an analogue of the two Hamming distances, which we might call the Hamming quality relations.

Let $\mathcal{O} \subseteq \mathcal{P}(U)$ be given now.

(Recall that we follow the preferential tradition, "smaller" will mean "better".)

$x \sim_s y :\Leftrightarrow \mathcal{O}(x) = \mathcal{O}(y)$,

$x \preceq_s y :\Leftrightarrow \mathcal{O}(y) \subseteq \mathcal{O}(x)$,

$x \prec_s y :\Leftrightarrow \mathcal{O}(y) \subset \mathcal{O}(x)$,

$x \sim_c y :\Leftrightarrow card(\mathcal{O}(x)) = card(\mathcal{O}(y))$,

$x \preceq_c y :\Leftrightarrow card(\mathcal{O}(y)) \leq card(\mathcal{O}(x))$,

$x \prec_c y :\Leftrightarrow card(\mathcal{O}(y)) < card(\mathcal{O}(x))$.

6.3.1.3 Distances

Note that we defined Hamming relations already in Section 5.2.3 (page 181), as announced in Section 1.5.4.4 (page 29).

Definition 6.3.7

Given $x, y \in \Sigma$, a set of sequences over an index set I, the Hamming distance comes in two flavours:

$d_s(x, y) := \{i \in I : x(i) \neq y(i)\}$, the set variant,

$d_c(x, y) := card(d_s(x, y))$, the counting variant.

We define $d_s(x, y) \leq d_s(x', y')$ iff $d_s(x, y) \subseteq d_s(x', y')$,

thus, $s-$distances are not always comparabel. Consequently, readers should be aware that d_s-values are *not* always comparable, even though $<$ and $\leq$ may suggest a linear order. We use these symbols to be in line with other distances.

There are straightforward generalizations of the counting variant:

We can also give different importance to different i in the counting variant, so e.g., $d_c(\langle x, x' \rangle, \langle y, y' \rangle)$ might be 1 if $x \neq y$ and $x' = y'$, but 2 if $x = y$ and $x' \neq y'$.

If the $x \in \Sigma$ may have more than 2 different values, then a varying individual distance may also reflect to the distances in Σ. So, (for any distance d) if $d(x(i), x'(i)) < d(x(i), x''(i))$, then (the rest being equal), we may have $d(x, x') < d(x, x'')$.

Fact 6.3.1

(1) If the $x \in \Sigma$ have only 2 values, say TRUE and FALSE, then $d_s(x, y) = \{i \in I : x(i) = TRUE\} \triangle \{i \in I : y(i) = TRUE\}$, where $\triangle$ is the symmetric set difference.

(2) d_c has the normal addition, set union takes the role of addition for d_s, $\emptyset$ takes the role of 0 for d_s, both are distances in the following sense:

(2.1) $d(x, y) = 0$ iff $x = y$,

(2.2) $d(x, y) = d(y, x)$,

(2.3) the triangle inequality holds, for the set variant in the form $d_s(x, z) \subseteq d_s(x, y) \cup d_s(y, z)$.

Proof

(2.3) If $i \notin d_s(x, y) \cup d_s(y, z)$, then $x(i) = y(i) = z(i)$, so $x(i) = z(i)$ and $i \notin d_s(x, z)$.

The others are trivial.

$\square$

Recall that the $\sigma \in \Sigma$ will often stand for a sequence of possibilities $O/\boldsymbol{C}(O)$ with $O \in \mathcal{O}$. Thus, the distance between two such sequences σ and σ' is the number or set of O, where σ codes being in O and σ' codes being in $\boldsymbol{C}(O)$, or vice versa.

Remark 6.3.2

If the $x(i)$ are equivalence classes, one has to be careful not to confound the distance between the classes and the resulting distance between elements of the classes, as two different elements in the same class have distance 0. So in Fact 6.3.1 (page 221) 2.1 only one direction holds.

Definition 6.3.8

(1) We can define for any distance d with some minimal requirements a notion of "between".

If the codomain of d has an ordering $\leq$, but no addition, we define:

$\langle x, y, z\rangle_d :\Leftrightarrow d(x, y) \leq d(x, z)$ and $d(y, z) \leq d(x, z)$.

If the codomain has a commutative addition, we define

$\langle x, y, z\rangle_d :\Leftrightarrow d(x, z) = d(x, y) + d(y, z)$ - in d_s + will be replaced by $\cup$, i.e.

$\langle x, y, z\rangle_s :\Leftrightarrow d(x, z) = d(x, y) \cup d(y, z)$.

For above two Hamming distances, we will write $\langle x, y, z\rangle_s$ and $\langle x, y, z\rangle_c$.

(2) We further define:

$[x, z]_d := \{y \in X : \langle x, y, x\rangle_d\}$ - where X is the set we work in.

We will write $[x, z]_s$ and $[x, z]_c$ when appropriate.

(3) For $x \in U$, $X \subseteq U$ set $x \parallel_d X := \{x' \in X : \neg\exists x'' \neq x' \in X.d(x, x') \geq d(x, x'')\}$.

Note that, if $X \neq \emptyset$, then $x \parallel X \neq \emptyset$.

We omit the index when this does not cause confusion. Again, when adequate, we write $\parallel_s$ and $\parallel_c$.

For problems with characterizing "between" see [Sch04].

Fact 6.3.3

(0) $\langle x, y, z\rangle_d \Leftrightarrow \langle z, y, x\rangle_d$.

Consider the situation of a set of sequences Σ, with $\sigma : I \to S$ for $\sigma \in \Sigma$

Let $A := A_{\sigma,\sigma''} := \{\sigma' : \forall i \in I(\sigma(i) = \sigma''(i) \to \sigma'(i) = \sigma(i) = \sigma''(i))\}$. Then

(1) If $\sigma' \in A$, then $d_s(\sigma, \sigma'') = d_s(\sigma, \sigma') \cup d_s(\sigma', \sigma'')$, so $\langle \sigma, \sigma', \sigma''\rangle_s$.

(2) If $\sigma' \in A$ and S consists of 2 elements (as in classical 2-valued logic), then $d_s(\sigma, \sigma')$ and $d_s(\sigma', \sigma'')$ are disjoint.

(3) $[\sigma, \sigma'']_s = A$.

(4) If, in addition, S consists of 2 elements, then $[\sigma, \sigma'']_c = A$.

Proof

(0) Trivial.

(1) "$\subseteq$" follows from Fact 6.3.1 (page 221), (2.3).

Conversely, if e.g. $i \in d_s(\sigma, \sigma')$, then by prerequisite $i \in d_s(\sigma, \sigma'')$.

(2) Let $i \in d_s(\sigma, \sigma') \cap d_s(\sigma', \sigma'')$, then $\sigma(i) \neq \sigma'(i)$ and $\sigma'(i) \neq \sigma''(i)$, but then by $card(S) = 2$ $\sigma(i) = \sigma''(i)$, but $\sigma' \in A$, *contradiction*.

We turn to (3) and (4):

If $\sigma' \notin A$, then there is i' such that $\sigma(i') = \sigma''(i') \neq \sigma'(i')$. On the other hand, for all i such that $\sigma(i) \neq \sigma''(i)$ $i \in d_s(\sigma,\sigma') \cup d_s(\sigma',\sigma'')$. Thus:

(3) By (1) $\sigma' \in A \Rightarrow \langle \sigma,\sigma',\sigma'' \rangle_s$. Suppose $\sigma' \notin A$, so there is i' such that $i' \in d_s(\sigma,\sigma') - d_s(\sigma,\sigma'')$, so $\langle \sigma,\sigma',\sigma'' \rangle_s$ cannot be.

(4) By (1) and (2) $\sigma' \in A \Rightarrow \langle \sigma,\sigma',\sigma'' \rangle_c$. Conversely, if $\sigma' \notin A$, then $card(d_s(\sigma,\sigma')) + card(d_s(\sigma',\sigma'')) \geq card(d_s(\sigma,\sigma'')) + 2$.

□

6.3.2 Obtaining such tools

We consider a set of sequences Σ, for $x \in \Sigma$ $x : I \to S$, I a finite index set, S some set. Often, S will be $\{0,1\}$, $x(i) = 1$ will mean that $x \in i$, when $I \subseteq \mathcal{P}(U)$ and $x \in U$. For abbreviation, we will call this (unsystematically, often context will tell) the $\in -$case . Often, I will be written $\mathcal{O}$, intuitively, $O \in \mathcal{O}$ is then an obligation, and $x(O) = 1$ means $x \in O$, or x "satisfies" the obligation O.

Definition 6.3.9

In the $\in -$case, set $\mathcal{O}(x) := \{O \in \mathcal{O} : x \in O\}$.

6.3.3 Additional requirements for neighbourhoods

Definition 6.3.10

Given any relation $\preceq$ (of quality), we say that $X \subseteq U$ is (downward) closed (with respect to $\preceq$) iff $\forall x \in X \forall y \in U (y \preceq x \Rightarrow y \in X)$.

(Warning, we follow the preferential tradition, "smaller" will mean "better".)

Fact 6.3.4

Let $\preceq$ be given.

(1) Let $D \subseteq U' \subseteq U''$, D closed in U'', then D is also closed in U'.

(2) Let $D \subseteq U' \subseteq U''$, D closed in U', U' closed in U'', then D is closed in U''.

(3) Let $D_i \subseteq U'$ be closed for all $i \in I$, then so are $\bigcup\{D_i : i \in I\}$ and $\bigcap\{D_i : i \in I\}$.

Proof

(1) Trivial.

(2) Let $x \in D \subseteq U'$, $x' \preceq x$, $x' \in U''$, then $x' \in U'$ by closure of U'', so $x' \in D$ by closure of U'.

(3) Trivial.

□

Definition 6.3.11

Given a quality relation $\prec$ between elements, and a distance d, we extend the quality relation to sets and define:

(1) $x \prec Y :\Leftrightarrow \forall y \in (x \parallel Y).x \prec y$. (The closest elements - i.e. there are no closer ones - of Y, seen from x, are less good than x.)

analogously $X \prec y :\Leftrightarrow \forall x \in (y \parallel X).x \prec y$

(2) $X \prec_l Y :\Leftrightarrow \forall x \in X.x \prec Y$ and $\forall y \in Y.X \prec y$ (X is locally better than Y).

When necessary, we will write $\prec_{l,s}$ or $\prec_{l,c}$ to distinguish the set from the counting variant.

For the next definition, we use the notion of size: $\nabla\phi$ iff for almost all ϕ holds i.e. the set of exceptions is small.

(3) $X \ll_l Y :\Leftrightarrow \nabla x \in X.x \prec Y$ and $\nabla y \in Y.X \prec y$.

We will likewise write $\ll_{l,s}$ etc.

This definition is supposed to capture quality difference under minimal change, the "ceteris paribus " idea: $X \prec_l \boldsymbol{C}X$ should hold for an obligation X. Minimal change is coded by $\parallel$, and "ceteris paribus" by minimal change.

Fact 6.3.5

If $X \prec_l \boldsymbol{C}X$, and $x \in U$ an optimal point (there is no better one), then $x \in X$.

Proof

If not, then take $x' \in X$ closest to x, this must be better than x, contradiction. $\Box$

Definition 6.3.12

Given a distance d, we define:

(1) Let $X \subseteq Y \subseteq U'$, then Y is a neighbourhood of X in U' iff

$\forall y \in Y \forall x \in X(x$ is closest to y among all x' with $x' \in X \Rightarrow [x,y] \cap U' \subseteq Y)$.

(Closest means that there are no closer ones.)

When we also have a quality relation $\prec$, we define:

(2) Let $X \subseteq Y \subseteq U'$, then Y is an improving neighbourhood of X in U' iff

$\forall y \in Y \forall x((x$ is closest to y among all x' with $x' \in X$ and $x' \preceq y) \Rightarrow [x,y] \cap U' \subseteq Y)$.

When necessary, we will have to say for (3) and (4) which variant, i.e. set or counting, we mean.

Definition 6.3.13

Given a Hamming distance and a Hamming relation, X is called a Hamming neighbourhood of the best cases iff for any $x \in X$ and y a best case with minimal distance from x, all elements between x and y are in X.

Fact 6.3.6

(1) If $X \subseteq X' \subseteq \Sigma$, and $d(x,y) = 0 \Rightarrow x = y$, then X and X' are Hamming neighbourhoods of X in X'.

(2) If $X \subseteq Y_j \subseteq X' \subseteq \Sigma$ for $j \in J$, and all Y_j are Hamming Neighbourhoods of X in X', then so are $\bigcup\{Y_j : j \in J\}$ and $\bigcap\{Y_j : j \in J\}$.

Proof

(1) is trivial (we need here that $d(x,y) = 0 \Rightarrow x = y$).

(2) Trivial.

$\square$

6.3.4 Connections between the various concepts

Fact 6.3.7

If x, y are models, then $[x,y] = M(\phi_{\{x,y\}})$. (See Definition 6.3.8 (page 222) and Definition 6.3.1 (page 219).)

Proof

$m \in [x,y] \Leftrightarrow \forall p(x \models p, y \models p \Rightarrow m \models p$ and $x \not\models p, y \not\models p \Rightarrow m \not\models p)$, $m \models \phi_{\{x,y\}} \Leftrightarrow m \models \bigwedge\{p : x(p) = y(p) = v\} \wedge \bigwedge\{\neg p : x(p) = y(p) = f\}$. $\square$

The requirement of closure causes a problem for the counting approach: Given e.g. two obligations O, O', then any two elements in just one obligation have the same quality, so if one is in, the other should be, too. But this prevents now any of the original obligations to have the desirable property of closure. In the counting case, we will obtain a ranked structure, where elements satisfy 0, 1, 2, etc. obligations, and we are unable to differentiate inside those layers. Moreover, the set variant seems to be closer to logic, where we do not count the propositional variables which hold in a model, but consider them individually. For these reasons, we will not pursue the counting approach as systematically as the set approach. One should, however, keep in mind that the counting variant gives a ranking relation of quality, as all qualities are comparable, and the set variant does not. A ranking seems to be appreciated sometimes in the literature, though we are not really sure why.

Of particular interest is the combination of d_s and $\preceq_s$ (d_c and $\preceq_c$) respectively - where by $\preceq_s$ we also mean $\prec_s$ and $\sim_s$, etc. We turn to this now.

Fact 6.3.8

We work in the $\in -$case.

(1) $x \preceq_s y \Rightarrow d_s(x,y) = \mathcal{O}(x) - \mathcal{O}(y)$

Let $a \prec_s b \prec_s c$. Then

(2) $d_s(a,b)$ and $d_s(b,c)$ are not comparable,

(3) $d_s(a,c) = d_s(a,b) \cup d_s(b,c)$, and thus $b \in [a,c]_s$.

This does not hold in the counting variant, as Example 6.3.1 (page 226) shows.

(4) Let $x \prec_s y$ and $x' \prec_s y$ with x, x' $\prec_s$ −incomparabel. Then $d_s(x,y)$ and $d_s(x',y)$ are incomparable.

(This does not hold in the counting variant, as then all distances are comparable.)

(5) If $x \prec_s z$, then for all $y \in [x,z]_s$ $x \preceq_s y \preceq_s z$.

Proof

(1) Trivial.

(2) We have $\mathcal{O}(c) \subset \mathcal{O}(b) \subset \mathcal{O}(a)$, so the results follows from (1).

(3) By definition of d_s and (1).

(4) x and x' are $\preceq_s$-incomparable, so there are $O \in \mathcal{O}(x) - \mathcal{O}(x')$, $O' \in \mathcal{O}(x') - \mathcal{O}(x)$.

As $x, x' \prec_s y$, $O, O' \notin \mathcal{O}(y)$, so $O \in d_s(x,y) - d_s(x',y)$, $O' \in d_s(x',y) - d_s(x,y)$.

(5) $x \prec_s z \Rightarrow \mathcal{O}(z) \subset \mathcal{O}(x)$, $d_s(x,z) = \mathcal{O}(x) - \mathcal{O}(z)$. By prerequisite $d_s(x,z) = d_s(x,y) \cup d_s(y,z)$. Suppose $x \not\preceq_s y$. Then there is $i \in \mathcal{O}(y) - \mathcal{O}(x) \subseteq d_s(x,y)$, so $i \notin \mathcal{O}(x) - \mathcal{O}(z) = d_s(x,z)$, *contradiction*.

Suppose $y \not\preceq_s z$. Then there is $i \in \mathcal{O}(z) - \mathcal{O}(y) \subseteq d_s(y,z)$, so $i \notin \mathcal{O}(x) - \mathcal{O}(z) = d_s(x,z)$, *contradiction*.

□

Example 6.3.1

In this and similar examples, we will use the model notation. Some propositional variables p, q, etc. are given, and models are described by $p\neg qr$, etc. Moreover, the propositional variables are the obligations, so in this example we have the obligations p, q, r.

Consider $x := \neg p \neg qr$, $y := pq\neg r$, $z := \neg p \neg q \neg r$. Then $y \prec_c x \prec_c z$, $d_c(x,y) = 3$, $d_c(x,z) = 1$, $d_c(z,y) = 2$, so $x \notin [y,z]_c$. □

Fact 6.3.9

Take the set version.

If $X \prec_{l,s} \boldsymbol{C}X$, then X is downward $\prec_s$-closed.

Proof

Suppose $X \prec_{l,s} \boldsymbol{C}X$, but X is not downward closed.

Case 1: There are $x \in X$, $y \notin X$, $y \sim_s x$. Then $y \in x \parallel_s \boldsymbol{C}X$, but $x \not\prec y$, *contradiction*.

Case 2: There are $x \in X$, $y \notin X$, $y \prec_s x$. By $X \prec_{l,s} \boldsymbol{C}X$, the elements in X closest to y must be better than y. Thus, there is $x' \prec_s y$, $x' \in X$, with minimal distance from y. But then $x' \prec_s y \prec_s x$, so $d_s(x', y)$ and $d_s(y, x)$ are incomparable by Fact 6.3.8 (page 225), so x is among those with minimal distance from y, so $X \prec_{l,s} \boldsymbol{C}X$ does not hold. $\square$

Example 6.3.2

We work with the set variant.

This example shows that $\preceq_s$ −closed does not imply $X \prec_{l,s} \boldsymbol{C}X$, even if X contains the best elements.

Let $\mathcal{O} := \{p, q, r, s\}$, $U' := \{x := p\neg q\neg r\neg s,\ y := \neg pq\neg r\neg s,\ x' := pqrs\}$, $X := \{x, x'\}$. x' is the best element of U', so X contains the best elements, and X is downward closed in U', as x and y are not comparable. $d_s(x, y) = \{p, q\}$, $d_s(x', y) = \{p, r, s\}$, so the distances from y are not comparable, so x is among the closest elements in X, seen from y, but $x \not\prec_s y$.

The lack of comparability is essential here, as the following Fact shows.

$\square$

We have, however, for the counting variant:

Fact 6.3.10

Consider the counting variant. Then

If X is downward closed, then $X \prec_{l,c} \boldsymbol{C}X$.

Proof

Take any $x \in X$, $y \notin X$. We have $y \preceq_c x$ or $x \prec_c y$, as any two elements are $\preceq_c$ −comparabel. $y \preceq_c x$ contradicts closure, so $x \prec_c y$, and $X \prec_{l,c} \boldsymbol{C}X$ holds trivially. $\square$

Example 6.3.3

Let $U' := \{x, x', y, y'\}$ with $x' := pqrs$, $y' := pqr\neg s$, $x := \neg p\neg qr\neg s$, $y := \neg p\neg q\neg r\neg s$.

Consider $X := \{x, x'\}$.

The counting version:

Then x' has quality 4 (the best), y' has quality 3, x has 1, y has 0.

$d_c(x', y') = 1$, $d_c(x, y) = 1$, $d_c(x, y') = 2$.

Then above “ceteris paribus” criterion is satisfied, as y' and x do not “see” each other, so $X \prec_{l,c}$ $\boldsymbol{C}X$.

But X is not downward closed, below $x \in X$ is a better element $y' \notin X$.

This seems an argument against X being an obligation.

The set version:

We still have $x' \prec_s y' \prec_s x \prec_s y$. As shown in Fact 6.3.8 (page 225), $d_s(x,y)$ (and also $d_s(x',y')$) and $d_s(x,y')$ are not comparable, so our argument collapses.

As a matter of fact, we have the result that the "ceteris paribus" criterion entails downward closure in the set variant, see Fact 6.3.9 (page 226).

□

In the following Section 6.3.4.1 (page 228) and Section 6.3.4.2 (page 230), we will assume a set $\mathcal{O}$ of obligations to be given. We define the relation $\prec := \prec_{\mathcal{O}}$ as described in Definition 6.3.6 (page 220), and the distance d is the Hamming distance based on $\mathcal{O}$, see Definition 6.3.7 (page 221).

We work here mostly in the set version, the $\in -$case, only in the final Section 6.3.4.3 (page 232), we will look at the counting case.

6.3.4.1 The not necessarily independent case

Example 6.3.4

Work in the set variant. We show that X $\preceq_s$-closed does not necessarily imply that X contains all $\preceq_s$-best elements.

Let $\mathcal{O} := \{p, q\}$, $U' := \{p\neg q, \neg pq\}$, then all elements of U' have best quality in U', $X := \{p\neg q\}$ is closed, but does not contain all best elements. □

Example 6.3.5

Work in the set variant. We show that X $\preceq_s$-closed does not necessarily imply that X is a neighbourhood of the best elements, even if X contains them.

Consider $x := pq\neg rstu$, $x' := \neg pqrs\neg t\neg u$, $x'' := p\neg qr\neg s\neg t\neg u$, $y := p\neg q\neg r\neg s\neg t\neg u$, $z := pq\neg r\neg s\neg t\neg u$. $U := \{x, x', x'', y, z\}$, the $\prec_s$ $-$best elements are x, x', x'', they are contained in $X := \{x, x', x'', z\}$. $d_s(z,x) = \{s,t,u\}$, $d_s(z,x') = \{p,r,s\}$, $d_s(z,x'') = \{q,r\}$, so x'' is one of the best elements closest to z. $d(z,y) = \{q\}$, $d(y,x'') = \{r\}$, so $[z,x''] = \{z, y, x''\}$, $y \notin X$, but X is downward closed. □

Fact 6.3.11

Work in the set variant.

Let $X \neq \emptyset$, X $\preceq_s$-closed. Then

(1) X does not necessarily contain all best elements.

Assume now that X contains, in addition, all best elements. Then

(2) $X \prec_{l,s} \boldsymbol{C}X$ does not necessarily hold.

(3) X is (ui).

(4) $X \in \mathcal{D}(\mathcal{O})$ does not necessarily hold.

(5) X is not necessarily a neighbourhood of the best elements.

(6) X is an improving neighbourhood of the best elements.

Proof

(1) See Example 6.3.4 (page 228)

(2) See Example 6.3.2 (page 227)

(3) If there is $m \in X$, $m \notin O$ for all $O \in \mathcal{O}$, then by closure $X = U$, take $\mathcal{O}_i := \emptyset$.

For $m \in X$ let $\mathcal{O}_m := \{O \in \mathcal{O} : m \in O\}$. Let $X' := \bigcup\{\bigcap \mathcal{O}_m : m \in X\}$.

$X \subseteq X'$: trivial, as $m \in X \to m \in \bigcap \mathcal{O}_m \subseteq X'$.

$X' \subseteq X$: Let $m' \in \bigcap \mathcal{O}_m$ for some $m \in X$. It suffices to show that $m' \preceq_s m$. $m' \in \bigcap \mathcal{O}_m = \bigcap\{O \in \mathcal{O} : m \in O\}$, so for all $O \in \mathcal{O}$ $(m \in O \to m' \in O)$.

(4) Consider Example 6.3.2 (page 227), let $dom(\delta) = \{r, s\}$, $\delta(r) = \delta(s) = 0$. Then $x, y \models \delta$, but $x' \not\models \delta$ and $x \in X$, $y \notin X$, but there is no $z \in X$, $z \models \delta$ and $z \prec y$, so $X \notin \mathcal{D}(\mathcal{O})$.

(5) See Example 6.3.5 (page 228).

(6) By Fact 6.3.8 (page 225), (5).

□

Fact 6.3.12

Work in the set variant

(1.1) $X \prec_{l,s} \boldsymbol{C}X$ implies that X is $\preceq_s$-closed.

(1.2) $X \prec_{l,s} \boldsymbol{C}X \Rightarrow X$ contains all best elements

(2.1) X is $(ui) \Rightarrow X$ is $\preceq_s$-closed.

(2.2) X is (ui) does not necessarily imply that X contains all $\preceq_s$-best elements.

(3.1) $X \in \mathcal{D}(\mathcal{O}) \Rightarrow X$ is $\preceq_s$-closed

(3.2) $X \in \mathcal{D}(\mathcal{O})$ implies that X contains all $\preceq_s$-best elements.

(4.1) X is an improving neighbourhood of the $\preceq_s$-best elements $\Rightarrow X$ is $\preceq_s$-closed.

(4.2) X is an improving neighbourhood of the best elements $\Rightarrow X$ contains all best elements.

Proof

(1.1) By Fact 6.3.9 (page 226).

(1.2) By Fact 6.3.5 (page 224).

(2.1) Let $O \in \mathcal{O}$, then O is downward closed (no $y \notin O$ can be better than $x \in O$). The rest follows from Fact 6.3.4 (page 223) (3).

(2.2) Consider Example 6.3.4 (page 228), p is (ui) (formed in $U!$), but $p \cap X$ does not contain $\neg pq$.

(3.1) Let $X \in \mathcal{D}(\mathcal{O})$, but let X not be closed. Thus, there are $m \in X$, $m' \preceq_s m$, $m' \notin X$.

Case 1: Suppose $m' \sim m$. Let $\delta_m : \mathcal{O} \to 2$, $\delta_m(O) = 1$ iff $m \in O$. Then $m, m' \models \delta_m$, and there cannot be any $m'' \models \delta_m$, $m'' \prec_s m'$, so $X \notin \mathcal{D}(\mathcal{O})$.

Case 2: $m' \prec_s m$. Let $\mathcal{O}' := \{O \in \mathcal{O} : m \in O \Leftrightarrow m' \in O\}$, $dom(\delta) = \mathcal{O}'$, $\delta(O) := 1$ iff $m \in O$ for $O \in \mathcal{O}'$. Then $m, m' \models \delta$. If there is $O \in \mathcal{O}$ such that $m' \notin O$, then by $m' \preceq_s m$ $m \notin O$, so $O \in \mathcal{O}'$. Thus for all $O \notin dom(\delta).m' \in O$. But then there is no $m'' \models \delta$, $m'' \prec_s m'$, as m' is already optimal among the n with $n \models \delta$.

(3.2) Suppose $X \in \mathcal{D}(\mathcal{O})$, $x' \in U - X$ is a best element, take $\delta := \emptyset$, $x \in X$. Then there must be $x'' \prec x'$, $x'' \in X$, but this is impossible as x' was best.

(4.1) By Fact 6.3.8 (page 225) , (4) all minimal elements have incomparabel distance. But if $z \preceq y$, $y \in X$, then either z is minimal or it is above a minimal element, with minimal distance from y, so $z \in X$ by Fact 6.3.8 (page 225) (3).

(4.2) Trivial.

□

6.3.4.2 The independent case

Assume now the system to be independent, i.e. all combinations of $\mathcal{O}$ are present.

Note that there is now only one minimal element, and the notions of Hamming neighbourhood of the best elements and improving Hamming neighbourhood of the best elements coincide.

Fact 6.3.13

Work in the set variant.

Let $X \neq \emptyset$, X $\preceq_s$-closed. Then

(1) X contains the best element.

(2) $X \prec_{l,s} \boldsymbol{C}X$

(3) X is (ui).

(4) $X \in \mathcal{D}(\mathcal{O})$

(5) X is a (improving) Hamming neighbourhood of the best elements.

Proof

(1) Trivial.

(2) Fix $x \in X$, let y be closest to x, $y \notin X$. Suppose $x \not\prec y$, then there must be $O \in \mathcal{O}$ such that $y \in O$, $x \notin O$. Choose y' such that y' is like y, only $y' \notin O$. If $y' \in X$, then by closure $y \in X$, so $y' \notin X$. But y' is closer to x than y is, *contradiction*.

Fix $y \in U-$X. Let x be closest to y, $x \in X$. Suppose $x \not\prec y$, then there is $O \in \mathcal{O}$ such that $y \in O$, $x \notin O$. Choose x' such that x' is like x, only $x' \in O$. By closure of X, $x' \in X$, but x' is closer to y than x is, *contradiction.*

(3) By Fact 6.3.11 (page 228) (3)

(4) Let X be closed, and $\mathcal{O}' \subseteq \mathcal{O}$, $\delta : \mathcal{O}' \to 2$, $m, m' \models \delta$, $m \in X$, $m' \notin X$. Let m'' be such that $m'' \models \delta$, and for all $O \in \mathcal{O} - dom(\delta)$ $m'' \in O$. This exists by independence. Then $m'' \preceq_s m'$, but also $m'' \preceq_s m$, so $m'' \in X$. Suppose $m'' \sim m'$, then $m' \preceq_s m''$, so $m' \in X$, contradiction, so $m'' \prec_s m'$.

(5) Trivial by (1), the remark preceding this Fact, and Fact 6.3.11 (page 228) (6).

Fact 6.3.14

Work in the set variant.

(1) $X \prec_{l,s} \boldsymbol{C}x \Rightarrow X$ is $\preceq_s$-closed,

(2) X is $(ui) \Rightarrow X$ is $\preceq_s$-closed,

(3) $X \in \mathcal{D}(\mathcal{O}) \Rightarrow X$ is $\preceq_s$-closed,

(4) X is a (improving) neighbourhood of the best elements $\Rightarrow X$ is $\preceq_s$-closed.

Proof

(1) Suppose there are $x \in X$, $y \in U-$X, $y \prec x$. Choose them with minimal distance. If $card(d_s(x,y)) > 1$, then there is z, $y \prec_s z \prec_s x$, $z \in X$ or $z \in U-$X, contradicting minimality. So $card(d_s(x,y)) = 1$. So y is among the closest elements of $U - X$ seen from x, but then by prerequisite $x \prec y$, *contradiction.*

(2) By Fact 6.3.12 (page 229) (2.1).

(3) By Fact 6.3.12 (page 229) (3.1).

(4) There is just one best element z, so if $x \in X$, then $[x,z]$ contains all y $y \prec x$ by Fact 6.3.8 (page 225) (3).

□

The $\mathcal{D}(\mathcal{O})$ condition seems to be adequate only for the independent situation, so we stop considering it now.

Fact 6.3.15

Let $X_i \subseteq U$, $i \in I$ a family of sets, we note the following about closure under unions and intersections:

(1) If the X_i are downward closed, then so are their unions and intersections.

(2) If the X_i are (ui), then so are their unions and intersections.

Proof

Trivial. □

We do not know whether $\prec_{l,s}$ is preserved under unions and intersections, it does not seem an easy problem.

Fact 6.3.16

(1) Being downward closed is preserved while going to subsets.

(2) Containing the best elements is not preserved (and thus neither the neighbourhood property).

(3) The $\mathcal{D}(\mathcal{O})$ property is not preserved.

(4) $\preceq_{l,s}$ is not preserved.

Proof

(4) Consider Example 6.3.3 (page 227), and eliminate y from U', then the closest to x not in X is y', which is better.

□

6.3.4.3 Remarks on the counting case

Remark 6.3.17

In the counting variant all qualities are comparabel. So if X is closed, it will contain all minimal elements.

Example 6.3.6

We measure distance by counting.

Consider $a := \neg p \neg q \neg r \neg s$, $b := \neg p \neg q \neg rs$, $c := \neg p \neg qr \neg s$, $d := pqr \neg s$, let $U := \{a, b, c, d\}$, $X := \{a, c, d\}$. d is the best element, $[a, d] = \{a, d, c\}$, so X is an improving Hamming neighbourhood, but $b \prec a$, so $X \not\prec_{l,c} \boldsymbol{C}X$.

□

Fact 6.3.18

We measure distances by counting.

$X \prec_{l,c} \boldsymbol{C}X$ does not necessarily imply that X is an improving Hamming neighbourhood of the best elements.

Proof

Consider Example 6.3.3 (page 227). There $X \prec_{l,c} \boldsymbol{C}X$. x' is the best element, and $y' \in [x', x]$, but $y' \notin X$. □

6.4 Neighbourhoods in deontic and default logic

6.4.1 Introduction

Deontic and default logic have very much in common. Both have a built-in quality relation, where situations which satisfy the deontic rules are better than those which do not, or closer to the normal case in the default situation.

They differ in the interpretation of the result. In default logic, we want to know what holds in the "best" or most normal situations, in deontic logic, we want to characterize the "good" situations, and avoid paradoxa like the Ross-paradox.

Note that our treatment concern only obligations and defaults without prerequisites, but this suffices for our purposes: to construct neighbourhood semantics for both. When we work with prerequisites, we have to consider the possibilities of branching into different "extensions", which is an independent problem.

We discussed MISE extensively in Section 2.3.2 (page 61), so it will not be necessary to repeat the presentation.

6.4.2 Two important examples for deontic logic

Example 6.4.1

The original version of the Ross paradox reads: If we have the obligation to post the letter, then we have the obligation to post or burn the letter. Implicit here is the background knowledge that burning the letter implies not to post it, and is even worse than not posting it.

We prefer a modified version, which works with two independent obligations: We have the obligation to post the letter, and we have the obligation to water the plants. We conclude by unrestricted weakening that we have the obligation to post the letter or *not* to water the plants. This is obvious nonsense.

Example 6.4.2

Normally, one should not offer a cigarette to someone, out of respect for his health. But the considerate assassin might do so nonetheless, on the cynical reasoning that the victim's health is going to suffer anyway:

(1) One should not kill, $\neg k$.

(2) One should not offer cigarettes, $\neg o$.

(3) The assassin should offer his victim a cigarette before killing him, if k, then o.

Here, globally, $\neg k$ and $\neg o$ is best, but among $k-$worlds, o is better than $\neg o$. The model ranking is $\neg k \wedge \neg o \prec \neg k \wedge o \prec k \wedge o \prec k \wedge \neg o$.

6.4.3 Neighbourhoods for deontic systems

A set $\mathcal{R}$ of deontic or default rules defines naturally quality and distance relations:

(1) A situation (model) m is better than a model m' iff m satisfies more rules than m' does. "More" can be defined by counting, or by the superset relation. In both cases, we will note the relation here by $\prec$. (See Definition 6.3.6 (page 220).)

(2) The distance between two models m, m' is the number - or the set - of rules satisfied by one, but not by the other. In both cases, we will note the distance here by d. Given a distance, we can define "between": a is between b and c iff $d(b,c) = d(b,a) + d(a,c)$ (in the case of sets, $+$ will be $\cup$). See Definition 6.3.7 (page 221) and Definition 6.3.8 (page 222).

We have here in each case two variants of Hamming relations or distances.

With these ideas, we can define "good" sets X in a number of ways:

Then, if $\mathcal{R}$ is a family of rules, and if x and x' are in the same subset $\mathcal{R}' \subseteq \mathcal{R}$ of rules, then a rule derived from $\mathcal{R}$ should not separate them. More precisely, if $x \in O \in \mathcal{R} \Leftrightarrow x' \in O \in \mathcal{R}$, and D is a derived rule, then $x \in D \Leftrightarrow x' \in D$.

We think that being closed is a desirable property for obligations: what is at least as good as one element in the obligation should be "in", too.

But it may be a matter of debate which of the different possible notions of neighbourhood should be chosen for a given deontic system. It seems, however, that we should use the characterization of neighbourhoods to describe acceptable situations. Thus, we determine the "best" situations, and all neighbourhoods of the best situations are reasonable approximations to the best situations, and can thus be considered "derived" from the original system of obligations.

Chapter 7

Conclusion and outlook

7.1 Conclusion

An important part of this book concerns the concept of independence. One of the authors described it - mostly as questions - as “homogenousness” in his first book, [Sch97-2]. But it took quite some time and detours to find a reasonable, and in hindsight obvious, answer.

7.1.1 Semantic and syntactic interpolation

We usually try to decompose a logical problem - often formulated in the language of this logic - into a semantical part, and then the translation to syntax. This has proven fruitful in the past, and does so here, too. The reason is that the semantical problems are often very different from the translation problems. The latter concern usually definabilty questions, which tend to be similar for various logical problems.

Here, we were able to see that *semantical* interpolation will always exist for monotonic or antitonic logics, but that the language and the operators may be too weak to define the interpolants syntactically. In contrast, semantical interpolants for non-monotonic logics need not always exist. We detail this now briefly.

7.1.2 Independence and interpolation for monotonic logic

Independence is closely related to (semantic) interpolation, as a matter of fact, in monotonic logic, the very definition of validity is based on independence, and guarantees semantic interpolation, also for many-valued logics, provided the order on the truth values is sufficiently strong, as we saw in Chapter 4 (page 125), wheras the expressive strength of the language determines whether we can $define$ the semantic interpolant (or interpolants), see the same chapter. We also saw that we often have an interval of interpolants, and the upper and lower limits are $universal$ interpolants in the following sense: they depend only on *one* formula, plus the set of propositional variables common to both formulas, but *not* on the second formula itself.

7.1.3 Independence and interpolation for non-monotonic logic

Perhaps the central chapter of the book is Chapter 5 (page 165), where we connect interpolation of non-monotonic logics to multiplicative laws about abstract size. The entrance to this is to see that logics like preferential logics define an abstract notion of size by considering the exceptional cases as forming a small subset of all cases, and, dually, the normal cases as a big subset of all cases. Then, laws for non-monotonic logics can be seen as laws about *addition* of small and big subsets, and about general well-behaviour of the notions of big and small. E.g., if X is a small subset of Y, and $Y \subseteq Y'$, then X should also be a small subset of Y'. These connections were investigated systematically first (to our knowledge) in [GS09a], see also [GS08f]. It was pointed out there that the rule of rational monotony does not fit well into laws about addition, and that it has to be seen rather as a rule about independence. It fits now well into our laws about *multiplication*, which express independence for non-monotonic logics. It is natural that the laws we need now for semantic interpolation are laws about multiplication, as we speak about products of model sets. Interpolation for non-monotonic logic has (at least) three different forms, where we may mix the non-monotonic consequence relation $\mid\sim$ with the classical relation $\vdash$. We saw that two variants are connected to such multiplicative laws, and, especially the weakest form has a translation to a very natural law about size. We can go on and relate these laws to natural laws about preferential relations, when the logic is preferential.

The problem of syntactic interpolation is the same as for the monotonic case.

These multiplicative laws about size have repercussions beyond interpolation, as they also say what should happen when we change the language, e.g., have a rule $\phi \mid\sim \psi$ in language L, and now chnage to a bigger language L', whether we can still expect $\phi \mid\sim \psi$ to hold. This seems a trivial problem, it is not, and somehow seems to have escaped attention so far.

7.1.4 Neighbourhood semantics

The concluding chapter of the book concerns neighbourhood semantics, see Chapter 6 (page 213). Such semantics are ubiquitous in non-classical logic, they can be found as systems of ever better sets in the limit version of preferential logics, as a semantics for deontic and default logics, for approximative logics, etc. We looked at the common points, how to define them, and what properties to require for them. This chapter should be seen as a toolbox, where one finds the tools to construct the semantics one needs for the particular case at hand.

7.2 Outlook

We think that further research should concern the dynamic aspects of reasoning, like iterated revision, revising one non-monotonic logic with another non-monotonic logic.

Moreover, it seems to us that any non-classical logic (which is not an extension of the former, like modal logic, but diverges in its results from classical logic) needs a $justification$, so such logics do not only consist of language, proof theory, and semantics, but of language, proof theory, semantics, *and* $justification$.

7.2.1 The dynamics of reasoning

So far, most work on non-monotonic and related logics concern one step in a reasoning process only. Notable exceptions are, e.g., [Spo88], and [DP94], [DP97].

It seems quite certain that there is no universal formalism, if, e.g., in a theory revision task, we are given ϕ, and then $\neg\phi$ as information, we can imagine situations where we should come back to the original state, and others, where this should not be the case.

So, the dynamics of reasoning need further investigation.

We took already some steps here, when we investigated generalized revision (see Chapter 3 (page 97), Section 3.4 (page 111)), as the results can be applied to revising one preferential logic with another one. (Usually, such logics will not have a natural ranked order, so traditional revision will not work.) But we need more than tools, we need satisfactory systems.

7.2.2 A revision of basic concepts of logic: justification

Some logics like inductive logics ("proving" a theory from a limited number of cases), non-monotonic logics, revision and update logics go beyond classical logic, they allow to derive formulas which cannot be derived in classical logic. Some might also be more modest, allowing less derivations, and some might be a mixture, e.g. approximative logics, allowing to derive some formulas which cannot be derived in classical logic, and not allowing to derive other formulas which can be derived in classical logic.

Let us call all those logics "bold logics".

Suppose that we agree that classical logic corresponds to "truth".

But then we need a $justification$ to do other logic than classical logic, as we know or suspect - or someone else knows or suspects - that our reasoning is in some cases false. (Let us suppose for simplicity that we know this erroneousness ourselves.)

Whatever this justification may be, we have now a fundamentally new situation.

Classical logic has language, proof theory, and semantics. Representation theorems say that the latter correspond. Non-monotonic logic also has (language and) proof theory, and semantics. But something is missing: the justification - which we do *not* need for classical logic, as we do not have any false reasoning to justify.

Thus,

- classical logic consists of
  (1) language (variables and operators),
  (2) proof theory,
  (3) semantics.
- bold logic consists of
  (1) language (variables and operators),
  (2) proof theory,

(3) semantics,

(4) justification.

If a bold logic has no justification - whatever that may be - it is just foolishness, and the bolder it is (the more it diverges from classical logic), the more foolish it is.

So let us consider justifications - in a far from exhaustive list.

(1) First, on the negative side, costs.

(1.1) A false result has a cost. This cost depends on the problem we try to solve. Suppose we have a case "man, blond". Classifying this case falsely as "man, black hair", has a different cost when we try to determine the amount of hair dyes to buy, and when we are on the lookout for a blond serial killer on the run.

(1.2) Calculating our bold logic has a cost, too (time and space). Usually, this will also depend on the case, the cost is not the same for all cases. E.g., let $T = p \vee (\neg p \vee q)$, then the cost to determine whether $m \models T$ is smaller for $p-$models, than for $\neg p-$models, as we have to check now in addition q.

In addition, there may be a global cost of calculation.

(2) Second, on the positive side, benefits:

(2.1) Classical logic also has its costs of calculation, similar to the above.

(2.2) In some cases, classical logic may not be strong enough to decide the case at hand. Hearing a vague noise in the jungle may not be enough to decide whether it is a lion or not, but we climb up the tree nonetheless. Bold logic allows us to avoid desaster, by precaution.

(2.3) Parsimony, elegance, promises to future elaboration, may also be considered benefits.

We can then say that a bold logic is justified, iff the benefits (summarized over all cases to consider) are at least as big as the costs (summarized over all cases to consider).

Diverging more from classical logic incurs a bigger cost, so the bolder a logic is, the stronger its benefits must be, otherwise it is not rational to choose it for reasoning.

When we look at preferential logic and its abstract semantics of "big" and "small" sets (see, e.g., [GS09a], [GS08f]), we can consider this semantics as being an *implicit* justification: The cases wrongly treated are together just a "small" or "unimportant" subset of all cases. (Note that this says nothing about the benefits.) But we want to make concepts clear, and explicit, however they may be treated in the end.

See also [GW08] for related reflections.

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